The Microscopic Spectral Density of the Dirac Operator derived from Gaussian Orthogonal and Symplectic Ensembles

Christian Hilmoine and Rune Niclasen

The Niels Bohr Institute
Blegdamsvej 17
DK-2100 Copenhagen, Denmark

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Abstract

The microscopic spectral correlations of the Dirac operator in Yang-Mills theories coupled to fermions in (2+1) dimensions can be related to three universality classes of Random Matrix Theory. In the microscopic limit the Orthogonal Ensemble (OE) corresponds to a theory with 2 colors and fermions in the fundamental representation and the Symplectic Ensemble (SE) corresponds to an arbitrary number of colors and fermions in the adjoint representation. Using a new method of Widom, we derive an expression for the two scalar kernels which through quaternion determinants give all spectral correlation functions in the Gaussian Orthogonal Ensemble (GOE) and in the the Gaussian Symplectic Ensemble (GSE) with all fermion masses equal to zero. The result for the GOE is valid for an arbitrary number of fermions while for the GSE we have results for an even number of fermions.

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1 Introduction

Random Matrix Theory has successfully been used to extract information about the spectral correlations of the Euclidean Dirac operator $D\!\!\!/ = \gamma_\mu(\partial_\mu + iA_\mu)$ eigenvalues in the low energy limit of Yang-Mills theories such as QCD. In $(3+1)$ dimensions and in the low energy limit the effective Yang-Mills partition function coincides with the partition function defined by the chiral Random Matrix Theory (χRMT) when the so-called microscopic limit is taken [1, 2, 3, 4]. The low-energy partition function describes the fermion mass dependence in the static limit and in a finite volume of space-time V [5] and is determined alone by global symmetries. The finite volume implies that we restrict to the case where only the low-lying excitations (the Goldstone modes) contribute to the field theory partition function, while the kinetic terms of the Lagrangian are neglected. In the microscopic limit we look at the Dirac operator spectra on the scale $\lambda = \mathcal{O}(V^{-1})$, which corresponds to a magnification of the spectra in the vicinity of $\lambda = 0$ on the scale V^{-1}. In case of spontaneous chiral symmetry breaking, reflected in a condensate different from zero, $\Sigma \neq 0$, the scale V^{-1} equals the average eigenvalue spacing. This follows from the Banks-Casher relation [6] $\Sigma = \pi \lim V^{-1}\rho(0)$, where $\rho(0)$ is the spectral density of the Dirac operator evaluated in origin and where first the thermodynamic and subsequently the chiral limit is taken. Thus we see that spontaneous chiral symmetry breaking is intimately related to the spectrum of the Dirac operator in origin.

In Random Matrix Theory no dynamical information is incorporated, only the global symmetries of the physical system are being used and therefore one studies the "universal" spectral correlations of the eigenvalues of the considered operator. Based on the work of Leutwyler and Smilga [7] it has been conjectured that the spectrum of the Dirac operator in QCD and similar theories is universal in the microscopic limit [1, 8]. This conjecture is supported by the fact that the sum rules derived by Leutwyler and Smilga, which involve inverse powers of the Dirac operator eigenvalues, can be derived from RMT [1, 2]. Thus, in this limit the microscopic spectral correlation functions (of which the microscopic spectral density is the most simple) of the Dirac operator can be derived from a much simpler Random Matrix Theory in which only the symmetries of the Dirac operator are the inputs. Depending on the representation of the gauge group $SU(N_c)$ and the number of colors N_c, the Dirac operator belongs to one of three universality classes, which in χRMT are represented by the orthogonal (χOE), the unitary (χUE) and the symplectic (χSE) ensemble [9]. For each theory the symmetries of the Dirac operator specifies one of these universality classes. In addition the chiral structure of the Dirac operator in all three theories is incorporated in χRMT, requiring a specific block structure of the matrices in the three ensembles χUE, χOE and χSE. See [2, 9] for a more detailed discussion.

Each of the three Yang-Mills theories in $(3+1)$ dimensions has an analogue in $(2+1)$ dimensions and the effective partition function of each theory has been showed to coincide with the partition function in each of the three universality classes defined by non-χ Random Matrix Theory (non-χRMT) [4, 10, 11]. In an odd number of space-time dimensions chiral symmetry does not exist. But in $(2+1)$ dimensions with an even number of flavors N_f it has been suggested that the spontaneous breakdown of flavor symmetry occurs, and this is the analogue of spontaneous chiral symmetry breaking in $(3+1)$ dimensions [4, 12]. Thus, the argumentation of the entire picture in $(3+1)$ dimensions has a parallel in $(2+1)$ dimensions [4, 10, 11]. Here we learn again that an order parameter Σ of the flavor symmetry breaking in $(2+1)$ dimensions is related to the spectral density of the Dirac operator, evaluated at zero, through a generalization of the Banks-Casher relation. The lack of chiral symmetry in $(2+1)$ dimensions is in the three ensembles non-χOE, non-χUE and non-χSE reflected in the lack of the chiral block structure of the matrices.

Thus in both $(3+1)$ and $(2+1)$ dimensions we have three types of Yang-Mills theories, defined by a choice of fermion colors N_c and a representation of the gauge group $SU(N_c)$, and the symmetries of the Dirac operator in each theory implies a specific structure of the Dirac matrix. Each theory is represented by a random matrix ensemble: in $(3+1)$ dimensions we have the three chiral (χ) ensembles and in $(2+1)$ dimensions the three non-chiral (non-χ) ensembles.

GAUGE GROUP	REPS.	DIRAC MATRIX	ENSEMBLE	β
SU(2)	Fundamental	Real	(non-)χOE	1
SU($N_c \geq 3$)	Fundamental	Complex	(non-)χUE	2
SU(N_c)	Adjoint	Quaternion real	(non-)χSE	4

Table 1: The table illustrates the classification, each ensemble is labelled by the Dyson index $\beta = 1, 2, 4$.

In all three Yang-Mills theories in $(3+1)$ dimensions, as well as in $(2+1)$ dimensions we can have an arbitrary number of flavors N_f of course. However, the coincidence of the ensembles $\beta = 1$ and $\beta = 4$ in $(2+1)$ dimensions with the corresponding effective field theory is only for *even* N_f. The relations to non-χGOE and non-χGSE of the two field theories was very recently derived by Magnea [10, 11]. In a theory with fermions in the fundamental representation of $SU(2)$ the following flavor symmetry breaking pattern causes the creation of the condensate Σ [10]:

$$Sp(2N_f) \rightarrow Sp(N_f) \times Sp(N_f). \tag{1}$$

In [11] the flavor symmetry breaking pattern

$$O(N_f) \rightarrow O(N_f/2) \times O(N_f/2), \tag{2}$$

is shown to be the one in a theory with N_c arbitrary and the adjoint representation of $SU(N_c)$. In this paper we want to derive the massless microscopic spectral density of the Dirac operator in these two field theories from the Random Matrix Theory of non-χOE and non-χSE.

The universality [13, 14] of the ensembles $\beta = 1$, $\beta = 2$ and $\beta = 4$ allows the choice of a Gaussian distribution (G) in these three ensembles, which is an advantage in view of the calculation of the spectral correlation functions. With the help of orthonormal polynomials all the microscopic spectral correlation functions have been derived in χGOE, χGUE and χGSE with massive fermions, see review in [15]. In non-χRMT, however, only the microscopic spectral correlation functions in non-χUE with an arbitrary number of massive fermions have been derived [4, 16, 17]. The two remaining universality classes are the non-χOE and non-χSE. In this paper we derive the kernels $S_N^{(\beta)}(x, y)$ for non-χGOE and non-χGSE with massless fermions. They determine all massless spectral correlation functions in these two ensembles. Specifically we derive the massless microscopic spectral density in the two ensembles. A direct verification of our results is possible through the generation of matrices distributed according to the probability distribution in the ensembles non-χGOE and non-χGSE. We also compare with the spectral sum rules recently derived in [10, 11].

The traditional method to derive spectral correlation functions in general orthogonal and symplectic ensembles with the use of polynomials is known as Dyson's quaternion matrix method [18]. In this method the kernels $S_N^{(\beta)}(x, y)$, $\beta = 1, 4$, are represented by special sums involving *skew*-orthonormal polynomials and the spectral correlation functions are determined by the quaternion determinant of a quaternion matrix given by $S_N^{(\beta)}(x, y)$, $\beta = 1, 4$. Widom and Tracy [24] have modified Dysons

quaternion matrix method, in the sense that the polynomials in the relevant kernels now can be chosen arbitrary. In [19] the relevant kernels for the ensembles $\beta = 1$ and $\beta = 4$ are given by *orthonormal* polynomials. The main content of section 3 in this work is to provide a simple recipe for how to derive the two kernels in an ensemble defined by a general weight function, only by the use of orthonormal polynomials. We avoid the actual proof [19] and focus on the construction of a helpful machinery to derive the needed kernels. In section 4 we use the recipe on the non-χGaussian ensembles $\beta = 1, 4$, defined in section 2, and thus derive an expression for the two kernels $S_N^{(\beta)}(x, y)$. Although the equivalences between the field theory partition function and the the partition functions in the two cases $\beta = 1, 4$, only are valid for even N_f, and thus the result only has interest for even N_f in Yang-Mills theory in $(2 + 1)$ dimensions [10, 11], the method gives us the spectral correlation functions also for odd N_f in the ensemble $\beta = 1$. In section 5 we present results for the microscopic spectral densities in the two ensembles and compare them with spectral sum rules and with Monte Carlo simulations done directly on random matrices.

2 The non-χGaussian ensembles

The non-χ random matrix model is defined by the partition function [4, 10, 11]

$$\mathcal{Z}_{N_f}^{(\beta)}(\mathcal{M}) = \int DT\ P_{N_f}^{(\beta)}(T) = \int DT\ \prod_{f=1}^{N_f} \det(iT + m_f) e^{-\frac{N\Sigma\beta}{4}\ \mathrm{Tr}\ \mathrm{V}(T^2)}, \tag{3}$$

where T is from an ensemble of hermitian $N \times N$ matrices and the integration is taken over the Haar measure DT. The Dyson index β has the value $\beta = 1$ for the orthogonal ensemble (non-χOE), $\beta = 2$ for the unitary ensemble (non-χUE) and $\beta = 4$ for the symplectic ensemble (non-χSE), which corresponds to real, complex and quaternion real matrix elements respectively. The matrix model is for a generic potential $V(T)$, but the basic assumption of universality justify the use of a Gaussian distribution, consistent with no additional input but the symmetries of the system. The condensate is $\Sigma \neq 0$, the parameter N_f is restricted to integers and the diagonalized mass matrix $\mathcal{M}$ is having N_f masses in the diagonal. The name "non-χ" is attached due to the inclusion of the determinant in (3), which makes the integrand not always positive and which makes these ensembles rather different from the usual orthogonal and symplectic ensembles near zero.
Putting all fermion masses to zero and deleting the determinant term ($N_f = 0$) makes the ensemble equivalent to the well known (classical) Gaussian ensemble and the replacement of the determinant with its absolute value (or restriction to even N_f) gives the generalized Gaussian ensemble [20]. The matrices in the *chiral* (χ) ensembles are rectangular in general and have a specific block structure, the former a result of the incorporation of the analogue of topological charge in the Random Matrix Theory and the latter corresponds to the choice of a representation of the Dirac matrix in a chiral basis. The lack of topological charge and chiral transformations in an odd number of space-time dimensions is in the non-χ RMT model (3) reflected in the quadratic matrices with no additional constraints, but the symmetries in each of the ensembles $\beta = 1, 2, 4$. This is exactly what separates the non-χ ensemble from the χ ensemble.

By decomposition of the matrices T one can easily transform to integration over the N eigenvalues $\{\lambda_k\}$ of T. Choosing the Gaussian distribution then gives the partition function

$$\mathcal{Z}_{N_f}^{(\beta)}(\mathcal{M}) = \int_{-\infty}^{\infty} \prod_{k=1}^{N} d\lambda_k \prod_{f=1}^{N_f/2} (\lambda_k^2 + m_f^2) e^{-\frac{N\beta\Sigma}{4}\lambda_k^2}\ |\Delta(\{\lambda_k\})|^\beta = \int_{-\infty}^{\infty} \prod_{k=1}^{N} d\lambda_k \omega_{N_f}^{(\beta)}(\lambda_k) |\Delta(\{\lambda_k\})|^\beta, \tag{4}$$

for even N_f, and for odd N_f we have an extra mass m which by the choice $m = 0$ gives [4]

$$\mathcal{Z}^{(\beta)}_{N_f}(\mathcal{M}) = \int_{-\infty}^{\infty} \prod_{k=1}^{N} d\lambda_k \lambda_k \prod_{f=1}^{(N_f-1)/2} (\lambda_k^2 + m_f^2) e^{-\frac{N\beta\Sigma}{4}\lambda_k^2} \; |\Delta(\{\lambda_k\})|^{\beta} = \int_{-\infty}^{\infty} \prod_{k=1}^{N} d\lambda_k \omega^{(\beta)}_{N_f}(\lambda_k) |\Delta(\{\lambda_k\})|^{\beta}, \quad (5)$$

were we have neglected all irrelevant overall factors. Here the function

$$\Delta(\{\lambda_i\}) \equiv \prod_{i<j}^{N} (\lambda_i - \lambda_j) \tag{6}$$

is the Vandermonde determinant and we have defined the *weight function* of the non-χ Gaussian ensemble

$$\omega^{(\beta)}_{N_f}(\lambda_k) \;\equiv \begin{cases} \displaystyle\prod_{f=1}^{N_f/2} (\lambda_k^2 + m_f^2) e^{-\frac{N\beta\Sigma}{4}\lambda_k^2} & \text{for } N_f \text{ even,} \\ \displaystyle\lambda_k \prod_{f=1}^{(N_f-1)/2} (\lambda_k^2 + m_f^2) e^{-\frac{N\beta\Sigma}{4}\lambda_k^2} & \text{for } N_f \text{ odd,} \end{cases} \tag{7}$$

on the support $\mathcal{I} =]-\infty, \infty[$. In the next section we outline the general method to derive the m-point spectral correlation function in a general ensemble defined by a weight function ω. Note that, in the massless case, i.e all $m_f = 0$, there is no separation between the two cases of even and odd N_f. Our goal is to derive an important function which determines the m-point spectral correlation functions in the two cases $\beta = 1, 4$ of the massless non-χGaussian ensembles.

3 The m-point correlation function

The spectrum of the ensemble defined by the general weight function ω on the support $\mathcal{I}$ is described by the *m-point correlation function*

$$R^{(\beta)}_m(x_1, \ldots, x_m) = \frac{1}{\mathcal{Z}^{(\beta)}_{N_f}} \frac{N!}{(N-m)!} \int_{\mathcal{I}} dx_{m+1} \ldots dx_N \prod_{j=1}^{N} \omega(x_j) \; |\Delta(\{x_i\})|^{\beta}. \tag{8}$$

This function gives the probability density that m of the eigenvalues, irrespective of their ordering, are located in infinitesimal neighborhoods of $x_1, \ldots, x_m$.
Now we will review how to derive the function $R^{(\beta)}_m$ with the help of *orthonormal polynomials* in all three ensembles $\beta = 1, 2, 4$. Define the functions

$$\varphi_i(x) \equiv p_i(x) \omega(x)^{1/2}, \quad i = 0, 1, 2, \ldots, \tag{9}$$

where $\{p_i(x)\}$ is the sequence of polynomials orthonormal with respect to the weight function $\omega(x)$ on $\mathcal{I}$. Thus the sequence $\{\varphi_i(x)\}$ consists of orthonormal functions on $\mathcal{I}$. Define the Hilbert space $\mathcal{H}$ spanned by the functions $\varphi_i(x)$, $i = 0, 1, 2, \ldots, (N-1)$. The projection operator K onto the space $\mathcal{H}$ has the kernel

$$K_N(x, y) \equiv \sum_{k=0}^{N-1} \varphi_k(x)\varphi_k(y) = \frac{a_N}{x-y} \begin{pmatrix} \varphi_N(x) & \varphi_{N-1}(x) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_N(y) \\ \varphi_{N-1}(y) \end{pmatrix}, \tag{10}$$

where the last equality follows from the important Christoffel-Darboux (see [21]) formula and $a_N = k_{N-1}/k_N$, k_N denoting the highest coefficient in $p_N(x)$. In the unitary ensemble ($\beta = 2$) the use of orthonormality and a simple rewriting of $|\Delta(\{\lambda_i\})|^2$ gives [18]

$$R_m^{(2)}(x_1, \ldots, x_m) = \det\left[K_N(x_i, x_j)\right]_{1\leq i,j\leq m}. \tag{11}$$

In the orthogonal and symplectic ensembles we have an analogous result for $R_m^{(\beta)}$, $\beta = 1, 4$. Here, however, the corresponding kernel $K_N^{(\beta)}$ is not a scalar, but a quaternion kernel and $R_m^{(\beta)}$ is represented by the quaternion determinant of quaternion matrix $[K_N(x_i, y_j)]_{1\leq i,j\leq m}$ [18]. Now, representing the quaternion kernels by their 2×2 matrix representations, whose entries all are given by one specific scalar kernel $S_N^{(\beta)}$, completes the analogous picture for the ensembles $\beta = 1, 4$: The matrix kernels

$$K_N^{(4)}(x, y) = \frac{1}{2}\begin{pmatrix} S_N^{(4)}(x, y) & S^{(4)}D(x, y) \\ IS_N^{(4)}(x, y) & S_N^{(4)}(y, x) \end{pmatrix}, \tag{12}$$

and

$$K_N^{(1)}(x, y) = \begin{pmatrix} S_N^{(1)}(x, y) & S^{(1)}D(x, y) \\ IS_N^{(1)}(x, y) - \varepsilon(x - y) & S_N^{(1)}(y, x) \end{pmatrix}, \tag{13}$$

determine the m-point correlation function

$$R_m^{(\beta)}(x_1, \ldots, x_m) = \mathrm{Q}\det\left[K_N^{(\beta)}(x_i, x_j)\right]_{1\leq i,j\leq m}, \qquad \beta = 1, 4. \tag{14}$$

Thus, the m-point correlation function is represented by the quaternion determinant of a $2m \times 2m$ matrix. Here $\varepsilon(x) = x/(2|x|)$ (the kernel of the operator ε), the scalar kernel $S_N^{(\beta)}(x, y)$ is given by certain sums of products involving the functions φ_n, and I and D are integration and differentiation operators, respectively. The operator $S^{(\beta)}$ has kernel $S_N^{(\beta)}(x, y)$, $S^{(\beta)}D(x, y)$ is the kernel of the operator $S^{(\beta)}D$ and $IS_N^{(\beta)}(x, y)$ is the kernel of $IS_N^{(\beta)}$. We see that the entire matrix kernel $K_N^{(\beta)}(x, y)$ is given by the scalar kernel $S_N^{(\beta)}(x, y)$, $\beta = 1, 4$. In the classical way derived by Dyson [22], and later generalized by Mehta and Mahoux [23], the polynomials in the functions φ_n, and therefore in $S_N^{(\beta)}(x, y)$, are chosen *skew*-orthonormal with respect to ω, which through an important theorem by Dyson gives the result (14). In [24], it has been shown that any choice of a family of polynomials leads to the same matrix kernel $K_N^{(\beta)}(x, y)$, $\beta = 1, 4$. The choice of skew-orthonormal polynomials leads apparently to the most simple $K_N^{(\beta)}(x, y)$. But the tedious work to derive skew-orthonormal polynomials and the lack of an relevant formula for this analogue of the Christoffel-Darboux formula (which is preferable when the scaling limit $N \to \infty$ is taken) asks for a representation of $S_N^{(\beta)}(x, y)$ in which the polynomials have well known properties and the sums involving them are more easy to deal with. This is precisely the result of [19]. For a weight function ω, with respect to which there exist orthonormal polynomials, and for which the function ω'/ω is a *rational* function, the two kernels $S_N^{(\beta)}(x, y)$, $\beta = 1, 4$, are given by [19] :

$$S_N^{(4)}(x, y) = K_{2N}(x, y) - \sum_{i>n,j=1}^{2n} [A_0 C_{00}^{-1} C_0]_{ij} \psi_i(x) \varepsilon\psi_j(y), \tag{15}$$

$$S_N^{(1)}(x, y) = K_N(x, y) - \sum_{i\leq n,j=1}^{2n} [AC(I - BAC)^{-1}]_{ji} \psi_i(x) \varepsilon\psi_j(y). \tag{16}$$

Both scalar kernels $S_N^{(\beta)}(x,y)$, $\beta=1,4$, equal a modified version of the scalar kernel (10) from the corresponding unitary ensemble, plus a linear combination of n functions. The number n equals the sum of orders of the poles of ω'/ω in the extended complex plane, where every endpoint of the support $\mathcal{I}$, where ω'/ω is analytic, is included as a simple pole. Thus, for the Gaussian ensemble ($\omega(x)=e^{-x^2}$, $\mathcal{I}=]-\infty,\infty[$) we have $n=1$ because of the simple pole in ∞ and for the Legendre ensemble ($\omega(x)=1$, $\mathcal{I}=[-1,1]$) $n=2$ because of the two endpoints. For the weight function ω the polynomials in (16) (the ensemble $\beta=1$) are orthonormal with respect to ω^2, meaning we must make the shift $\omega^{1/2}\rightarrow\omega$ in (9). In the function (15), the involved polynomials are orthonormal with respect to ω, as in the ensemble with $\beta=2$, but here the shift $N\rightarrow 2N$ must be taken. The relevant Hilbert space for each ensemble is thus

$$\begin{aligned} \beta=1 &: \qquad \mathcal{H}=\mathrm{span}\{\varphi_0,\varphi_1,\ldots,\varphi_{N-1}\}, \qquad \text{where} \qquad \varphi_i(x)=p_i(x)\omega(x), \\ \beta=4 &: \qquad \mathcal{H}=\mathrm{span}\{\varphi_0,\varphi_1,\ldots,\varphi_{2N-1}\}, \qquad \text{where} \qquad \varphi_i(x)=p_i(x)\omega(x)^{1/2}, \end{aligned} \tag{17}$$

and $p_i(x)$ is a polynomial of degree i. The functions $\psi_i(x)$, $i=1,2,\ldots,2n$, are determined from the two functions $\varphi_N(x)$, $\varphi_{N-1}(x)$ ($N\rightarrow 2N$ for $\beta=4$ and $\omega^{1/2}\rightarrow\omega$ for $\beta=1$), and the poles of ω'/ω. Once we have determined the $2n$ linear independent functions, that is precisely $\psi_i\in\mathcal{H}$, $i=1,2,\ldots,n$, and $\psi_i\in\mathcal{H}^{\perp}$, $i=(n+1),\ldots,2n$, the matrices A, B and C (and A_0, C_{00} and C_0, which just are block parts of A and C) can be derived from these. In the next section we describe the procedure in great detail.
The results (15) and (16) have their origin in identities for each operator $S^{(\beta)}$, $\beta=1,4$, which are valid for a general ω, telling that $S^{(\beta)}$ equals the projection operator K onto the Hilbert space $\mathcal{H}$, plus a correction. This correction is in each ensemble $\beta=1,4$ given by the operators K, D, and ε, and in case the operator $[D,K]$ has finite rank, i.e independent of N, the kernel of this correction constitute finitely many terms. When ω'/ω is a rational function this is precisely the case, and the choice of orthonormal polynomials with respect to ω and ω^2 for $\beta=4$ and $\beta=1$ respectively, leads to (15) and (16).
In the proceeding section we construct a recipe for how to derive the n corrections on the right hand sides of (15) and (16).

3.1 Deriving the kernels $S_N^{(\beta)}(x_i,x_j)$, $\beta=1,4$, with the help of orthonormal polynomials

For a semi-classical weight function ω, i.e ω'/ω is a rational function, the following procedure [19] leads to the derivation of the two associated kernels $S_N^{(\beta)}(x_i,x_j)$, $\beta=1,4$. We restrict to weight functions on the form

$$\omega(x)=e^{-V(x)}, \tag{18}$$

fulfilling

$$\lim_{x\rightarrow\partial\mathcal{I}}\omega(x)=0, \tag{19}$$

where $V(x)$ is continuously differentiable in the interior of $\mathcal{I}$. (For certain weight functions the condition (19) is unnecessary [26].) The n correction terms in each of the two ensembles are determined by the poles of ω'/ω and the two functions

$$\beta=1 \quad : \qquad \varphi_i(x)=p_i(x)\omega(x), \quad i=(N-1),\ N \tag{20}$$

$$\beta=4 \quad : \qquad \varphi_i(x)=p_i(x)\omega(x)^{\frac{1}{2}}, \quad i=(2N-1),\ 2N, \tag{21}$$

where the polynomials p_i are chosen orthonormal with respect to $\omega^2(x) = e^{-2V(x)}$ and $\omega(x) = e^{-V(x)}$, respectively. The following steps give the right hand side of (15) and (16), where the specific functions given above, belonging to each ensemble, are used. In the ensemble $\beta = 1$ we assume that N is even. With the purpose to work with both ensembles at the same time we make the shift $2N \to N$ for $\beta = 4$. Therefore, in the end of the procedure we must remember to double up the matrix dimension $N \to 2N$ for $\beta = 4$. Here is the recipe summarizing the general method [19] for obtaining the two kernels $S_N^{(\beta)}(x,y)$, $\beta = 1,4$:

(1) At first the main contribution to $S_N^{(\beta)}(x,y)$, $\beta = 1,4$, is determined. For each ensemble $\beta = 1,4$, the kernel $K_N(x,y)$ is constructed from the associated two functions, (20) or (21), through (10).

(2) Using the definition

$$U(x,y) = c^{(\beta)}\,\frac{V'(x) - V'(y)}{x-y}, \qquad c^{(\beta)} = \begin{cases} 1 & \text{for} \quad \beta = 4 \\ 2 & \text{for} \quad \beta = 1, \end{cases}$$

calculate the four functions

$$A_N(x) \equiv a_N \int_{\mathcal{I}} dy\ \varphi_N(x)\varphi_{N-1}(y)U(x,y), \tag{22}$$

$$B_N(x) \equiv a_N \int_{\mathcal{I}} dy\ \varphi_N^2(y)U(x,y), \tag{23}$$

$$A(x) \equiv A_N(x) - \frac{c^{(\beta)}}{2}V'(x), \qquad C(x) \equiv \frac{a_N}{a_{N-1}}B_{N-1}(x), \tag{24}$$

for each $\beta = 1,4$. (The $c^{(\beta=1)} = 2$ comes from the factor 2 in the exponent of the weight function ω^2.) The kernel of the operator $[D,K]$ is $[D,K](x,y)$ and it is given by [1]

$$[D,K](x,y) = a_N \begin{pmatrix} \varphi_N(x) & \varphi_{N-1}(x) \end{pmatrix} \begin{pmatrix} \frac{C(x)-C(y)}{x-y} & \frac{A(x)-A(y)}{x-y} \\ \frac{A(x)-A(y)}{x-y} & \frac{B_N(x)-B_N(y)}{x-y} \end{pmatrix} \begin{pmatrix} \varphi_N(y) \\ \varphi_{N-1}(y) \end{pmatrix}. \tag{26}$$

Here the central matrix is vital for the derivation of the specific matrix A (not to be confused with the function $A(x)$) later on.

(3) Next, the number $n = n_\infty + \sum_{j=1}^{N_p} n_{x_j}$ is calculated. The orders $\{n_{x_j}\}$ of the N_p poles $\{x_j\}$ of the function ω'/ω are added. A pole at infinity with order n_∞ is included as well as the endpoints of $\mathcal{I}$ in which ω is analytic, the latter are counted as simple poles ($n_{x_j} = 1$). (Notice that $(\omega^2)'/\omega^2$, relevant for $\beta = 1$, has the same poles with same orders as ω'/ω.)

(4) By writing out the power series of the rational functions $A(x)$, $B_N(x)$ and $C(x)$ in (26) one can easily verify that the kernel $[D,K](x,y)$ can be represented by a linear combination of the $2n$ functions

$$x^k\varphi_{N-1}(x), \qquad x^k\varphi_N(x), \qquad (0 \le k < n_\infty), \tag{27}$$

[1] This is easily derived with the help of [26] :

$$\begin{pmatrix} \varphi_N'(x) \\ \varphi_{N-1}'(x) \end{pmatrix} = \begin{pmatrix} A(x) & B_N(x) \\ -C(x) & -A(x) \end{pmatrix} \begin{pmatrix} \varphi_N(x) \\ \varphi_{N-1}(x) \end{pmatrix}, \tag{25}$$

and the representation (10) of $K_N(x,y)$.

$$(x-x_i)^{-k-1}\varphi_{N-1}(x), \qquad (x-x_i)^{-k-1}\varphi_N(x), \qquad (0 \le k < n_{x_i}). \tag{28}$$

In addition, the space, $\mathcal{G}$, spanned by these $2n$ functions has a subspace of dimension n contained in $\mathcal{H}$ and another subspace of dimension n contained in $\mathcal{H}^\perp$. Therefore, we now determine $2n$ linearly independent functions $\psi_i(x) \in \mathcal{H}$, $i = 1, 2, \ldots, n$, and $\psi_i(x) \in \mathcal{H}^\perp$, $i = (n+1), \ldots, 2n$. It is easily shown that all the functions in $\mathcal{G}$ are orthogonal to all functions contained in the space $\mathcal{H}/\mathcal{S}$, where

$$\mathcal{S} = \mathrm{span}\{\varphi_{N-k}, \varphi_l\}, \qquad 0 < k \le n_\infty, \quad 0 \le k < (n - n_\infty), \tag{29}$$

is a subspace of $\mathcal{H}$ and has dimension n. This implies that the n functions $\psi_i \in \mathcal{H}^\perp$ are given by the demand $\psi_i \in \mathcal{S}^\perp$, simplifying the derivation of the functions.
We have

$$[D, K](x, y) = \sum_{i,j=1}^{2n} A_{ij}\psi_i(x)\psi_j(y), \tag{30}$$

where the $2n \times 2n$ matrix A now must be derived through (26) by the specific choice of the functions $\psi_i(x)$, $i = 1, 2, \ldots, 2n$. The matrix A is symmetric (because K is symmetric and D is anti-symmetric) and satisfies

$$A_{ij} = 0 \quad \text{if} \quad i, j \le n \quad \text{or} \quad i, j > n, \tag{31}$$

which reduces the work. There are not other restrictions on the choice of the functions ψ_i, but the linear independence, and the matrix A is uniquely determined through (26) by the given choice.

(5) The $2n \times 2n$ matrix B defined by the inner product

$$B_{ij} \equiv (\varepsilon\psi_i, \psi_j) = (\int_{\mathcal{I}} dy \varepsilon(x-y)\psi_i(y)\ ,\ \psi_j) = \int_{\mathcal{I}}\int_{\mathcal{I}} dx\ dy\ \varepsilon(x-y)\psi_i(y)\psi_j(x), \tag{32}$$

must now be calculated. The fact that $\varepsilon(x) = -\varepsilon(-x)$ implies

$$B_{ij} = -B_{ji}, \qquad B_{ii} = 0, \tag{33}$$

meaning that we only have to determine $n(2n-1)$ matrix elements. Note that the elements are numbers depending on the parameters N and β. In general the result for $S_N^{(\beta)}(x, y)$ in the scaling limit $N \to \infty$ has the highest interest. If an asymptotic relation for the orthonormal polynomials (and thus for ψ_i) is well known, it is therefore relevant to examine whether or not it is legal to put in the asymptotic relations when this limit is taken in (32). It is certainly preferable to interchange the integration and the limit $N \to \infty$ in (32), the justification of which is given by Lebesgue's Majorant Theorem. The general procedure is valid for all (even) N, though, and in the next steps we work with the general matrix B, with no reference to the actual calculation of the matrix elements.

(6) The matrices on the right hand sides of (15) and (16) are given by the $2n \times 2n$ matrices J and C

$$J_{ij} \equiv \delta_{ij} - (\delta_{ij})|_{i,j>n}, \quad 0 \le i, j \le 2n, \tag{34}$$

and

$$C \equiv J + BA. \tag{35}$$

For instance for $n = 1$ we have

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{36}$$

The matrix A_0 is defined by the $2n \times n$ matrix produced by deleting the last n columns in A and the $n \times 2n$ matrix C_0 is defined by the matrix produced by deleting the last n rows in C. In addition we have the $n \times n$ matrix C_{00} given by deleting the last n rows and the last n columns in C. We need only the n last rows of the matrix

$$A_0 C_{00}^{-1} C_0, \tag{37}$$

and the n first columns of

$$AC(I - BAC)^{-1}. \tag{38}$$

(7) The functions $\varepsilon\psi_i(x) = \int_{\mathcal{I}} \varepsilon(x-y)\psi_i(y)$ are ingredients in the n correction terms in (15) and in (16), as well as in the matrix elements $B_{ij} = (\varepsilon\psi_i, \psi_j)$. If allowed, it is highly preferable to put in the asymptotic relation for $\psi_i(x)$ in the scaling limit $N \to \infty$. We must solve the integrals explicitly if this is not allowed.

(8) The results of points (1), (4), (6) and (7) are collected to construct the right hand side of (15) and (16). For $\beta = 4$ we must remember the replacement $N \to 2N$ everywhere, giving the right hand side of (15).

4 The kernels $S_N^{(\beta)}(x_i, x_j)$

In this section we use the recipe of last section to derive the two kernels (15) and (16) in the the massless non-χ Gaussian ensemble, defined by the choice $m_f = 0$ in the weight function (7).
For all $m_f = 0$ we have the weight function

$$\omega_{N_f}^{(\beta)}(x) = x^{N_f} e^{-cx^2}, \qquad \beta = 1, 4, \tag{39}$$

on the support $\mathcal{I} =]-\infty, \infty[$, where $c = \frac{N\Sigma\beta}{4}$. The *spectral density* $\rho(x)$ is the 1-point correlation function and from (8) we have [4]

$$\rho(-x) = (-1)^{NN_f} \rho(x). \tag{40}$$

We will always take N even making the spectral density an even function, also when N_f is odd. For *even* N_f the polynomials [20]

$$H_{2m}^{N_f}(x) = (-1)^m (\bar{h}_m^{N_f})^{-\frac{1}{2}} c^{\frac{1+N_f}{4}} L_m^{\frac{N_f-1}{2}}(cx^2), \qquad \bar{h}_m^{N_f} = \frac{\Gamma(\frac{N_f}{2} + m + \frac{1}{2})}{m!}, \tag{41}$$

$$H_{2n+1}^{N_f}(x) = (-1)^n (\bar{h}_n^{N_f})^{-\frac{1}{2}} c^{\frac{3+N_f}{4}}\, x\, L_n^{\frac{N_f+1}{2}}(cx^2), \qquad \bar{h}_n^{N_f} = \frac{\Gamma(\frac{N_f}{2} + n + \frac{3}{2})}{n!}, \tag{42}$$

are orthonormal with respect to (39) on $\mathcal{I} =]-\infty, \infty[$. Here the functions $L_n^\alpha(x)$ are the generalized Laguerre polynomials (which are orthonormal with respect to $\omega(x) = x^\alpha e^{-x}$, $\alpha > -1$, on $\mathcal{I} =]0, \infty[$), with the co efficient of x^n given by

$$k_n = (-1)^n / n!. \tag{43}$$

We call the polynomials (41) and (42) generalized Hermite polynomials. Notice that these polynomials satisfy

$$H_{2m}^{N_f}(-x) = H_{2m}^{N_f}(x), \quad H_{2n+1}^{N_f}(-x) = -H_{2n+1}^{N_f}(x), \tag{44}$$

and

$$H_N^{N_f}(x) = \frac{1}{x} H_{N+1}^{N_f-2}(x), \tag{45}$$

the latter property is easily derived from a basic relation between Laguerre polynomials. Letting $\bar{k}_i^{N_f}$ be the coefficient of the highest power in $H_i^{N_f}(x)$ (and remembering that N is even) implies

$$a_N = \frac{\bar{k}_{N-1}}{\bar{k}_N} = \frac{(-1)^{\frac{N-2}{2}}\ k_{\frac{N-2}{2}}}{(-1)^{\frac{N}{2}}\ k_{\frac{N}{2}}} \frac{(\bar{h}_{N-1}^{N_f})^{-\frac{1}{2}}}{(\bar{h}_N^{N_f})^{-\frac{1}{2}}} \frac{c^{\frac{N-2}{2}} c^{\frac{3+N_f}{4}}}{c^{\frac{N}{2}} c^{\frac{1+N_f}{4}}} = (\frac{N}{2})^{\frac{1}{2}} c^{-\frac{1}{2}}, \tag{46}$$

where $k_i^{N_f}$ is the coefficient (43). The generalized Hermite polynomials are relevant for all three cases $\beta = 1, 2, 4$ and even N_f. For $\beta = 2, 4$ and even N_f we must use the polynomials (41) and (42) which are orthonormal with respect to (39), but for odd N_f we cannot use the orthonormal polynomial method described earlier, due to the fact that there does not exist orthonormal polynomials with respect to an odd weight function defined on an even interval [21]. For $\beta = 1$, however, we must use the orthonormal polynomials with respect to $\omega_{N_f}^{(\beta=1)}(x)^2 = x^{2N_f} e^{-2cx^2}$, implying that we can choose N_f both even and odd in this case. The relevant polynomials are given (41) and (42) with the replacements $N_f \to 2N_f$ and $c \to 2c$, which we will carry out at the end of derivations. Thus, in both cases $\beta = 1, 4$ the number N_f $(2N_f)$ is even. According to the prescription of the case $\beta = 4$ we will do the replacement $N \to 2N$ at the end of the derivations.

Following the recipe described in last section we now construct the kernels $S_N^{(\beta)}(x_i, x_j)$, $\beta = 1, 4$, defined by the weight function (39).
With the polynomials $H_i^{N_f}(x)$, given by (41) and (42), and $\omega_{N_f}^{(\beta)}(x)$, given by (39), we have

$$\varphi_i(x) = H_i^{N_f}(x)\ \omega_{N_f}^{(\beta)}(x)^{\frac{1}{2}}, \tag{47}$$

$i = 0, 1, \ldots,$ and $\mathcal{H} = \text{span}\{\varphi_0, \varphi_1, \ldots, \varphi_{N-1}\}$. In each ensemble $\beta = 1, 4$ we now have the kernel $K_N(x, y)$, through (10).
Because N_f is even in both ensembles we have $\omega_{N_f}^{(\beta)}(x) = e^{-cx^2 + N_f \ln|x|}$. It follows then, that

$$U(x, y) = \frac{V'(x) - V'(y)}{x - y} = \frac{N_f}{xy} + \frac{(cx^2)' - (cy^2)'}{x - y} = \frac{N_f}{xy} + 2c. \tag{48}$$

The polynomial $H_{N-1}^{N_f}(x)$ does not have a constant term, see (42), and therefore the function (22) vanishes. We have

$$\begin{aligned} A_N(x) = a_N \int_{-\infty}^{\infty} dy\ H_N^{N_f}(y) H_{N-1}^{N_f}(y) (\frac{N_f}{xy} + 2c)\ y^{N_f} \exp(-cy^2) \\ = \quad a_N \frac{N_f}{x} \int_{-\infty}^{\infty} dy\ H_N^{N_f}(y) p_{N-2}(y)\ y^{N_f} \exp(-cy^2) = 0, \end{aligned} \tag{49}$$

where $p_{N-2}(y) = H_{N-1}^{N_f}(y)/y$. The function (23) equals

$$B_N(x) = a_N \int_{-\infty}^{\infty} dy\ (H_N^{N_f}(y))^2 (\frac{N_f}{xy} + 2c)\ y^{N_f} \exp(-cy^2). \tag{50}$$

It follows from the orthonormality properties that the integral of the second term in (50) equals $2ca_N$. The integral of the first term gives

$$\begin{aligned} & a_N \frac{N_f}{x} \int_{-\infty}^{\infty} dy \ (H_N^{N_f}(y))^2 \ y^{N_f-1} \exp(-cy^2) \\ = \quad & a_N \frac{N_f}{x} \frac{1}{N_f} y^{N_f} (H_N^{N_f}(y))^2 \exp(-cy^2)|_{-\infty}^{\infty} - a_N \frac{N_f}{x} \frac{1}{N_f} \int_{-\infty}^{\infty} dy \ y^{N_f} \Big(H_N^{N_f}(y) \exp(-cy^2) \Big)' \\ = \quad & -a_N \frac{1}{x} \int_{-\infty}^{\infty} dy \ y^{N_f} [2H_N^{N_f}(y) H_N'^{N_f}(y) - 2yc(H_N^{N_f}(y))^2] \exp(-cy^2) = 0, \end{aligned} \tag{51}$$

implying that

$$B_N(x) = 2ca_N = \sqrt{2N} c^{\frac{1}{2}}. \tag{52}$$

We then have the functions

$$\begin{aligned} A(x) = -\frac{1}{2} V'(x) = -\frac{1}{2}(2cx - \frac{N_f}{x}) = -cx + \frac{N_f}{2x}, \qquad & B_N(x) = 2ca_N = \sqrt{2N} c^{\frac{1}{2}}, \\ C(x) = \frac{a_N}{a_{N-1}} B_{N-1} = 2ca_N = B_N(x) = \sqrt{2N} c^{\frac{1}{2}}. & \end{aligned} \tag{53}$$

Now we have the matrix in (26), making it possible to determine the matrix A after the choice of a number $2n$ of functions $\psi_i(x)$, see (30).
The function

$$\frac{\frac{d}{dx}\omega_{N_f}^{(\beta)}(x)}{\omega_{N_f}^{(\beta)}(x)} = \frac{N_f}{x} - 2cx, \tag{54}$$

has a pole at $x = 0$ and at $x = \infty$, with orders $n_0 = 1$ and $n_\infty = 1$ respectively. Thus we have $n = n_\infty + n_0 = 2$ meaning that we have to calculate two correction terms.
We must find $n = 2$ linearly independent functions $\psi_1, \psi_2 \in \mathcal{H}$ and 2 linearly independent $\psi_3, \psi_4 \in \mathcal{H}^\perp$. These functions must be written as linear combinations of the functions (27) and (28), with only the case $k = 0$, and each ψ_i, $i = 1, 2, 3, 4$, is therefore a linear combination of the four functions

$$\varphi_{N-1}(x), \qquad \varphi_N(x), \qquad \frac{\varphi_{N-1}(x)}{x}, \qquad \frac{\varphi_N(x)}{x}. \tag{55}$$

The function $\frac{\varphi_{N-1}(x)}{x}$ is a polynomial of degree $(N-2)$ times $\omega_{N_f}^{(\beta)}(x)^{1/2}$ and thus it follows immediately that

$$\psi_1(x) \equiv \varphi_{N-1}(x), \quad \psi_2(x) \equiv \frac{\varphi_{N-1}(x)}{x} \in \mathcal{H}. \tag{56}$$

In addition, with the help of orthogonality it is easily shown that

$$\psi_3(x) \equiv \varphi_N(x), \quad \psi_4(x) \equiv \frac{\varphi_N(x)}{x} \in \mathcal{H}^\perp. \tag{57}$$

The requirements are simply that $\psi_3 \ \psi_4 \notin \mathcal{H}$ and $\psi_3, \psi_4 \perp \varphi_0$ and $\psi_3, \psi_4 \perp \varphi_{N-1}$ (see (29)), which follows from orthogonality.
With the choices (56) and (57) we now determine the matrix A through (26), (53) og (30). Inserting (53), (56) and (57), into (26) gives

$$[D, K](x, y) = -a_N \Big[c\varphi_N(x)\varphi_{N-1}(y) + c\varphi_{N-1}(x)\varphi_N(y) + \frac{N_f}{2xy}(\varphi_N(x)\varphi_{N-1}(y) + \varphi_{N-1}(x)\varphi_N(y)) \Big], \tag{58}$$

A Comparison with (30) gives the matrix elements of the 4×4-matrix A. We get

$$A = \begin{pmatrix} 0 & 0 & -a_N c & 0 \\ 0 & 0 & 0 & -a_N \frac{N_f}{2} \\ -a_N c & 0 & 0 & 0 \\ 0 & -a_N \frac{N_f}{2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -(\frac{N}{2})^{1/2} c^{\frac{1}{2}} & 0 \\ 0 & 0 & 0 & -(\frac{N}{2})^{1/2} c^{-\frac{1}{2}} \frac{N_f}{2} \\ -(\frac{N}{2})^{1/2} c^{\frac{1}{2}} & 0 & 0 & 0 \\ 0 & -(\frac{N}{2})^{1/2} c^{-\frac{1}{2}} \frac{N_f}{2} & 0 & 0 \end{pmatrix}, \tag{59}$$

which is symmetric and fulfills equation (31).
Because of (33) the 4×4-matrix B is given by the elements

$$B_{12} = (\varepsilon\psi_1, \psi_2), \quad B_{13} = (\varepsilon\psi_1, \psi_3), \quad B_{14} = \quad (\varepsilon\psi_1, \psi_4) \quad \text{and} \tag{60}$$

$$B_{23} = (\varepsilon\psi_2, \psi_3), \quad B_{14} = (\varepsilon\psi_1, \psi_4), \quad B_{34} = (\varepsilon\psi_3, \psi_4). \tag{61}$$

The generalized Hermite polynomials $H_i^{N_f}(x)$ are either even or odd functions and the same is true for the square root of the weight function $\omega_{N_f}^{(\beta)}(x)^{1/2}$ (because N_f is even), and thus the functions $\psi_i(x)$, $i = 1, 2, 3, 4$, are even or odd. A general calculation of the inner product gives $(\varepsilon f, g) = 0$, whenever f and g have same parity. From this it follows that

$$B_{14} = (\varepsilon\psi_1, \psi_4) = B_{23} = (\varepsilon\psi_2, \psi_3) = 0, \tag{62}$$

which together with (33) results in

$$B = \begin{pmatrix} 0 & (\varepsilon\psi_1, \psi_2) & (\varepsilon\psi_1, \psi_3) & 0 \\ -(\varepsilon\psi_1, \psi_2) & 0 & 0 & (\varepsilon\psi_2, \psi_4) \\ -(\varepsilon\psi_1, \psi_3) & 0 & 0 & (\varepsilon\psi_3, \psi_4) \\ 0 & -(\varepsilon\psi_2, \psi_4) & -(\varepsilon\psi_3, \psi_4) & 0 \end{pmatrix}. \tag{63}$$

We have

$$B_{ij} = (\varepsilon\psi_i, \psi_j) = \int_{-\infty}^{\infty} dx \psi_j(x) \int_{-\infty}^{\infty} dy \varepsilon(x-y)\psi_i(y) = \int_{-\infty}^{\infty} dx \psi_j(x) \frac{1}{2} \Big\{ \Big(\int_{-\infty}^{-x} dy + \int_{-x}^{x} dy \Big) - \int_{x}^{a} dy \Big\} \psi_i(y), \tag{64}$$

from which it follows that

$$B_{ij} = 2 \int_0^{\infty} dx \ \psi_j(x) \int_0^x dy \ \psi_i(y), \tag{65}$$

whenever ψ_i is an even function. Notice that in every element, $B_{ij} \neq 0$, one of the functions ψ_i or ψ_j is always even while the other is odd, which together with the feature $B_{ij} = -B_{ji}$ implies that the absolute value of all matrix elements B_{ij} can be written on the form (65). Leaving these matrix elements unsolved for a while, we now focus on the structure of the kernels $S_N^{(\beta)}(x, y)$, $\beta = 1, 4$.
Having the matrices A, B and $(n = 2)$

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad C = J + BA, \qquad A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ -(\frac{N}{2})^{1/2} c^{\frac{1}{2}} & 0 \\ 0 & -(\frac{N}{2})^{1/2} c^{-\frac{1}{2}} \frac{\alpha}{2} \end{pmatrix}, \tag{66}$$

and so on, the two kernels are given by

$$S_N^{(4)}(x, y) = K_N(x, y) - \psi_3(x) \sum_{l=1}^{4} [A_0 C_{00}^{-1} C_0]_{3l} \ \varepsilon\psi_l(y) - \psi_4(x) \sum_{k=1}^{4} [A_0 C_{00}^{-1} C_0]_{4k} \ \varepsilon\psi_k(y), \tag{67}$$

$$S_N^{(1)}(x,y) = K_N(x,y) - \psi_1(x)\sum_{l=1}^{4}[AC(I-BAC)^{-1}]_{l1}\ \varepsilon\psi_l(y) - \psi_2(x)\sum_{k=1}^{4}[AC(I-BAC)^{-1}]_{k2}\ \varepsilon\psi_k(y), \tag{68}$$

where

$$\varepsilon\psi_i(x) = \frac{1}{2}\Big\{\int_{-\infty}^{x}dy - \int_{x}^{\infty}dy\Big\}\psi_i(y) = \frac{1}{2}\Big\{\int_{-\infty}^{-x}dy + \int_{-x}^{x}dy - \int_{x}^{\infty}dy\Big\}\psi_i(y), \tag{69}$$

$i = 1, 2, 3, 4$. If ψ_i is an even function we see that

$$\varepsilon\psi_i(x) = \int_0^x dy\psi_i(y), \tag{70}$$

while if ψ_i is odd then

$$\varepsilon\psi_i(x) = -\int_x^{\infty} dy\psi_i(y). \tag{71}$$

Now we turn to the troublesome integrals in the kernels. The explicit expressions for the four functions $\psi_i(x)$ are given by $\varphi_j(x) = H_j^{(\beta)}(x)\omega_{N_f}^{(\beta)}(x)^{1/2}$, (56) and (57) and are

$$\begin{aligned}
\psi_1(x) &= (-1)^{\frac{N}{2}-1}\Big[\frac{(\frac{N}{2}-1)!}{\Gamma(\frac{N}{2}+\frac{N_f+1}{2})}\Big]^{\frac{1}{2}}\ c^{\frac{3+N_f}{4}}\ L_{\frac{N}{2}-1}^{\frac{N_f+1}{2}}(cx^2)x^{\frac{N_f}{2}+1}e^{-c\frac{x^2}{2}},\\
\psi_2(x) &= \frac{\psi_1(x)}{x},\\
\psi_3(x) &= (-1)^{\frac{N}{2}}\Big[\frac{(\frac{N}{2})!}{\Gamma(\frac{N}{2}+\frac{N_f+1}{2})}\Big]^{\frac{1}{2}}\ c^{\frac{1+N_f}{4}}\ L_{\frac{N}{2}}^{\frac{N_f-1}{2}}(cx^2)x^{\frac{N_f}{2}}e^{-c\frac{x^2}{2}},\\
\psi_4(x) &= \frac{\psi_3(x)}{x}.
\end{aligned} \tag{72}$$

For relatively large N it is clear that an actual calculation of the two kernels $S_N^{(\beta)}(x,y)$ becomes quite involved due to the presence of the integrals (65), (70) and (71). Actually, it is the microscopic limit of the kernel

$$S_N^{(\beta)}(x,y)_s \equiv \lim_{N\to\infty}\frac{1}{N\Sigma}S_N^{(\beta)}(\frac{x}{N\Sigma},\frac{y}{N\Sigma}), \qquad \beta = 1,4, \tag{73}$$

giving all the microscopic correlations, which has physical interest. Therefore it is necessary to determine whether it is allowed to substitute an asymptotic relation for the polynomials in the integrals when the microscopic limit is taken. If the answer to this is positive, we can get rid of the N dependence and derive a closed analytical expression for the microscopic kernel (73). With the help of Lebesgue's Majorant Theorem this question is pursued and answered in Appendix A. For (70), of course, the use of an asymptotic relation for ψ_i is allowed when the microscopic limit is taken, because the integral is finite for all N. We must check that the second equality in the following equation

$$\lim_{N\to\infty}N^{\lambda}B_{ij} = \lim_{N\to\infty}N^{\lambda}2\int_0^{\infty}dx\ \psi_j(x)\int_0^x dy\ \psi_i(y) = 2\int_0^{\infty}dx\lim_{N\to\infty}N^{\lambda}\ \psi_j(x)\int_0^x dy\ \psi_i(y), \tag{74}$$

as well as the last equality in (see (71))

$$\lim_{N\to\infty}N^{\delta}\varepsilon\psi_i(\frac{x}{N}) = -\lim_{N\to\infty}N^{\delta}\int_{\frac{x}{N}}^{\infty}dy\psi_i(y) = -\lim_{N\to\infty}N^{\delta}\frac{1}{N}\int_x^{\infty}du\psi_i(\frac{u}{N}) = -\int_x^{\infty}du\lim_{N\to\infty}N^{\delta-1}\psi_i(\frac{u}{N}), \tag{75}$$

are valid. Here we assume that the power of N^λ and N^δ have the exact values needed to make the number $N^\lambda B_{ij}$ and the function $N^{\delta-1}\psi_i(x/N)$ convergent for $N\to\infty$. This is a consequence of the assumption that the microscopic kernel (73) is finite.
Now let us examine (74) for $i=1$ and $j=2$. After two substitutions we have

$$B_{12} = C_N \int_0^\infty dz \;\; L_{\frac{N}{2}-1}^{\frac{N_f+1}{2}}(2z) z^{\frac{1}{2}(\frac{N_f}{2}-1)} e^{-z} \int_0^z du \;\; L_{\frac{N}{2}-1}^{\frac{N_f+1}{2}}(2u) u^{\frac{N_f}{4}} e^{-u}, \tag{76}$$

where all irrelevant factors have been collected in C_N. (We remember that (65) only holds when ψ_i is an even function, meaning that $N_f/2$ is odd in case of $i=1$ (see (72)). But the fact $B_{ij}=-B_{ji}$ implies that all B_{ij} can be represented on the form (65), when we keep track of the sign.) Using the result of Appendix A.2 we find that (74) does not hold for B_{12}. From (122) and point (2) at the end of A.2, this is a consequence of

$$-\Big(\frac{N_f+1}{4}+\frac{1}{4}-\frac{N_f}{4}\Big) = -\frac{1}{2} \geq -1, \tag{77}$$

and

$$-\Big(\frac{N_f+1}{4}+\frac{1}{4}-(\frac{N_f}{4}-\frac{1}{2})\Big) = -1 \geq -1. \tag{78}$$

A similar treatment of the explicit expressions for the three other matrix elements B_{13}, B_{24} and B_{34} gives that the corresponding equation (74) does *not* hold for any of them. An undesirable consequence is that we have to solve all four integrals B_{12}, B_{13}, B_{24} and B_{34} before enlarging N in the kernels (67) and (68).
Turning to the question of the validity of the last equality in (75) we must use the result of Appendix A.1. Consider for instance ψ_1 and assume that this function is odd, that is $((N_f/2)+1)$ is odd. Then from (72) and (71) we have

$$\begin{aligned} \varepsilon\psi_1(\frac{x}{N}) &= -\frac{1}{N}\int_x^\infty du\; \psi_1(\frac{u}{N}) \propto -\frac{1}{N}\int_x^\infty du\; L_{\frac{N}{2}-1}^{\frac{N_f+1}{2}}(c\frac{u^2}{N^2})\Big(\frac{u}{N}\Big)^{\frac{N_f}{2}+1} e^{-c\frac{u^2}{2N^2}} \\ &\propto -\int_x^\infty dt\; L_{\frac{N}{2}-1}^{\frac{N_f+1}{2}}(2t) t^{\frac{N_f}{4}} e^{-t}, \end{aligned} \tag{79}$$

where all irrelevant factors have been skipped. From (109) and (120) it now follows that (75) is *not* valid for $i=1$. This is caused by the fact

$$\frac{N_f+1}{4}-\frac{N_f}{4} = \frac{1}{4} \leq \frac{3}{4}. \tag{80}$$

Considering the functions $\varepsilon\psi_i(x)$, $i=2,3,4$, with $\psi_i(x)$ odd (which of course is not possible for all functions at the same time), an analogue argumentation gives the same conclusion. When the function $\psi_i(x)$, $i=1,2,3,4$, is odd we therefore have to solve the integral $\varepsilon\psi_i(x)$ to make an exact calculation of the kernels (68) and (67) possible (within reasonable time) for large N.

We have learned that for finite N a calculation of B_{12}, B_{13}, B_{24} and B_{34} and an analytical expression for the four functions $\varepsilon\psi_i(x)$, $i=1,2,3,4$, are needed to make the results (68) and (67) usable. This requirement has been met in Appendix B.

In Appendix B.1 we derive an expression for the function

$$\mathcal{E}_{[\bar\alpha,\bar\beta,n]}(x) \equiv \varepsilon\; L_n^{\bar\alpha}(x^2) x^{\bar\beta} e^{-\frac{x^2}{2}}, \tag{81}$$

for integers $n, \bar\beta$ and $\alpha > -1$. The functions $\varepsilon\psi_i(x)$, $i = 1,2,3,4$, belong to this class of functions and it is easily derived that (see (72))

$$\varepsilon\psi_1(x) = k_1 c^{-\frac{1}{4}} \mathcal{E}_{[\frac{N_f+1}{2},\ \frac{N_f}{2}+1,\ \frac{N}{2}-1]}(c^{\frac{1}{2}}x), \tag{82}$$

$$\varepsilon\psi_2(x) = k_2 c^{\frac{1}{4}} \mathcal{E}_{[\frac{N_f+1}{2},\ \frac{N_f}{2},\ \frac{N}{2}-1]}(c^{\frac{1}{2}}x), \tag{83}$$

$$\varepsilon\psi_3(x) = k_3 c^{-\frac{1}{4}} \mathcal{E}_{[\frac{N_f-1}{2},\ \frac{N_f}{2},\ \frac{N}{2}]}(c^{\frac{1}{2}}x), \tag{84}$$

$$\varepsilon\psi_4(x) = k_4 c^{\frac{1}{4}} \mathcal{E}_{[\frac{N_f-1}{2},\ \frac{N_f}{2}-1,\ \frac{N}{2}]}(c^{\frac{1}{2}}x), \tag{85}$$

with the coefficients

$$\begin{aligned} k_1 &= (-1)^{\frac{N}{2}-1}\Big[\frac{(\frac{N}{2}-1)!}{\Gamma(\frac{N}{2}+\frac{N_f+1}{2})}\Big]^{\frac{1}{2}}, \\ k_2 &= k_1, \\ k_3 &= (-1)^{\frac{N}{2}}\Big[\frac{(\frac{N}{2})!}{\Gamma(\frac{N}{2}+\frac{N_f+1}{2})}\Big]^{\frac{1}{2}}, \\ k_4 &= k_3. \end{aligned} \tag{86}$$

For each ψ_i the associated function $\mathcal{E}$ above is given by (144). From (72) it is clear that if $N_f/2$ is even, then ψ_1 and ψ_4 are odd functions and ψ_2 and ψ_3 are even functions. For odd $N_f/2$, of course, the situation is the other way around. Thus for each kernel (68) and (67) our derived expression will split into two parts, one for even $N_f/2$ and one for odd $N_f/2$.
In Appendix B.2 the result for the function $\mathcal{E}$ is used to calculate

$$\mathcal{B}_{ij} \equiv (\varepsilon\phi_i, \phi_j) = \int_{-\infty}^{\infty} dx\ \phi_j(x)\ \int_{-\infty}^{\infty} dy\ \varepsilon(x-y)\phi_i(y) \tag{87}$$

where

$$\phi_j(x) = L_m^{\alpha}(x^2)x^{\beta}e^{-\frac{x^2}{2}} \quad \text{og} \quad \phi_i(x) = L_n^{\bar\alpha}(x^2)x^{\bar\beta}e^{-\frac{x^2}{2}}, \tag{88}$$

and $n, m, \beta, \bar\beta$ are integers, fulfilling certain conditions. We have always an odd $\bar\beta$, meaning that ϕ_i is odd. By keeping track of a sign we can always, with the help of $B_{ij} = -B_{ji}$ as described earlier, secure that the function in the inner integral of the double integral $B_{ij} = (\varepsilon\psi_i, \psi_j)$ is odd (or even). Thus the four relevant numbers B_{ij} are contained in the derived expressions for (87) in Appendix B.2. Putting the functions (72) into (87), and remembering the mentioned possible change of sign when we secure that the odd function comes in the inner integral of B_{ij}, gives the following result for the four matrix elements of B :

$$B_{12} = (-1)^{\frac{N_f}{2}} k_1^2 \mathcal{B}_{12}, \tag{89}$$

$$B_{13} = (-1)^{\frac{N_f}{2}} c^{-\frac{1}{2}} k_1 k_3 \mathcal{B}_{13}, \tag{90}$$

$$B_{24} = (-1)^{\frac{N_f}{2}+1} c^{\frac{1}{2}} k_2 k_4 \mathcal{B}_{24}, \tag{91}$$

$$B_{34} = (-1)^{\frac{N_f}{2}+1} (k_3)^2 \mathcal{B}_{34}, \tag{92}$$

where the numbers $\mathcal{B}_{ij}$ are found with the help of table 2 and 3.

even $\frac{N_f}{2}$	given by eq.	with parameters
$\mathcal{B}_{12}$	(159), *without* the (+)-term	$\alpha, \bar{\alpha} = \frac{N_f+1}{2},\ \beta = \frac{N_f}{2},\ \bar{\beta} = \frac{N_f}{2}+1,\ n = m = \frac{N}{2}-1$
$\mathcal{B}_{13}$	(159), *without* the (+)-term	$\alpha = \frac{N_f-1}{2},\ \bar{\alpha} = \frac{N_f+1}{2},\ \beta = \frac{N_f}{2},\ \bar{\beta} = \frac{N_f}{2}+1,\ m = \frac{N}{2},\ n = \frac{N}{2}-1$
$\mathcal{B}_{24}$	(164)	$\alpha = \frac{N_f+1}{2},\ \bar{\alpha} = \frac{N_f-1}{2},\ \beta = \frac{N_f}{2},\ \bar{\beta} = \frac{N_f}{2}-1,\ m = \frac{N}{2}-1,\ n = \frac{N}{2}$
$\mathcal{B}_{34}$	(159), *without* the (+)-term	$\alpha = \bar{\alpha} = \frac{N_f-1}{2},\ \beta = \frac{N_f}{2}-1,\ \bar{\beta} = \frac{N_f}{2},\ m = n = \frac{N}{2}$

Table 2: A recipe giving $\mathcal{B}_{ij}$ for *even* $N_f/2$.

odd $\frac{N_f}{2}$	given by eq.	with parameters
$\mathcal{B}_{12}$	(159), *without* the (+)-term	$\alpha, \bar{\alpha} = \frac{N_f+1}{2},\ \beta = \frac{N_f}{2}+1,\ \bar{\beta} = \frac{N_f}{2},\ n = m = \frac{N}{2}-1$
$\mathcal{B}_{13}$	(159), *including* the (+)-term	$\alpha = \frac{N_f+1}{2},\ \bar{\alpha} = \frac{N_f-1}{2},\ \beta = \frac{N_f}{2}+1,\ \bar{\beta} = \frac{N_f}{2},\ m = \frac{N}{2}-1,\ n = \frac{N}{2}$
$\mathcal{B}_{24}$	(159), *without* the (+)-term	$\alpha = \frac{N_f-1}{2},\ \bar{\alpha} = \frac{N_f+1}{2},\ \beta = \frac{N_f}{2}-1,\ \bar{\beta} = \frac{N_f}{2},\ m = \frac{N}{2},\ n = \frac{N}{2}-1$
$\mathcal{B}_{34}$	(159), *without* the (+)-term	$\alpha = \bar{\alpha} = \frac{N_f-1}{2},\ \beta = \frac{N_f}{2},\ \bar{\beta} = \frac{N_f}{2}-1,\ m = n = \frac{N}{2}$

Table 3: A recipe giving $\mathcal{B}_{ij}$ for *odd* $N_f/2$.

Having calculated the four matrix elements of B and the four functions $\varepsilon\psi_i$ we can now construct the two kernels (68) and (67) of the non-χGaussian ensemble. The proceeding points summarize the construction of the kernels :

(1) Through equation (10), the kernel $K_N(x, y)$ is given by (47) with the associated comments.

(2) Assume that $(\frac{N_f}{2})$ is *even* (odd). Then it follows from (72), that ψ_1 and ψ_4 are odd (even), and ψ_2 and ψ_3 are even (odd). The functions $\varepsilon\psi_1$, $\varepsilon\psi_2$, $\varepsilon\psi_3$ and $\varepsilon\psi_4$ are given by (82), (83), (84) and (85), respectively, with $\mathcal{E}$ given by (144) both for even and odd ψ_i.

(3) The matrix elements B_{12}, B_{13}, B_{24} and B_{34} depend on whether $N_f/2$ is even or odd. These are given by (89), (90), (91) and (92), with $\mathcal{B}_{ij}$ given by tables 2 and 3. Matrix B is given by (63).

(4) The matrix

$$C = J + BA \tag{93}$$

is given by (59), (63) and (66). We need the matrices

$$\begin{aligned} A_0 C_{00}^{-1} C_0 \quad &\text{for} \quad \beta = 4, \quad \text{and} \\ AC(I - BAC)^{-1} \quad &\text{for} \quad \beta = 1. \end{aligned} \tag{94}$$

(5) Before putting all pieces together in the construction of (68) and (67) we remember the replacements

$$\begin{aligned} \beta = 1 \quad &: \quad c \to 2c \quad \text{and} \quad N_f \to 2N_f, \\ \beta = 4 \quad &: \quad N \to 2N, \end{aligned} \tag{95}$$

in all parts above and in the functions (72). In the two ensembles $\beta = 1, 4$ we have $c = \frac{N\Sigma^2\beta}{4}$.

The five points above leads to two formulas for each $S_N^{(\beta)}(x, y)$, $\beta = 1, 4$: one for even N_f and one for odd N_f for $\beta = 1$, and one for even $N_f/2$ and one for odd $N_f/2$ for $\beta = 4$.

5 The microscopic spectral density

In the microscopic limit the spectral correlations between the eigenvalues of the Dirac operator in the two Yang-Mills theories mentioned in the introduction can be derived from the corresponding symplectic or orthogonal non-χGaussian ensembles. In this section we present our results for our numerical expressions for the microscopic spectral density.

From (14) it follows that the 1-point correlation function, that is the spectral density, is given by

$$\rho^{(\beta)}(x) \equiv R_N^{(\beta)}(x) = S_N^{(\beta)}(x,x), \qquad \beta = 1,4. \tag{96}$$

The machinery from the last section gives both $S_N^{(\beta)}(x,y)$, $\beta = 1,4$, and we therefore have in principle all correlation functions and especially the microscopic spectral density in the cases $\beta = 1$ and $\beta = 4$ of the non-χGaussian ensemble. The *microscopic* spectral density is defined by

$$\rho_s^{(\beta)}(x) \equiv \lim_{N\to\infty} \frac{1}{N}\rho^{(\beta)}(\frac{x}{N}), \qquad \beta = 1,4, \tag{97}$$

and this is of physical interest. We have not yet extracted the N dependence of our results for the B_{ij} (derived in appendix B.2) and for the functions $\varepsilon\psi_i$ (derived in appendix B.1) in the microscopic limit and we therefore have not an exact analytical result for $\rho_s^{(\beta)}(x)$, $\beta = 1,4$. However, we can easily present plots of the scaled spectral density

$$\rho_N^{(\beta)}(x) \equiv \frac{1}{N}\rho^{(\beta)}(\frac{x}{N}) = \frac{1}{N}S_N^{(\beta)}(\frac{x}{N},\frac{x}{N}), \qquad \beta = 1,4, \tag{98}$$

which, of course, converges towards $\rho_s^{(\beta)}(x)$ for large N.

Due to the presence of the Vandermonde determinant in the probability distribution one expects that the spectrum becomes more rigid when the parameter β is increased (see (8)). Thus we naturally expect the scaled spectral density $\rho_N^{(1)}(x)$ in the ensemble non-χGOE to be flat compared to $\rho_N^{(2)}$ in non-χGUE, and that $\rho_N^{(4)}$ in non-χGSE is the most oscillating function of the three of them. These features are of course valid for all values of N.
In both ensembles $\beta = 1,4$ the kernel $S_N^{(\beta)}(x,y)$ is equal to a kernel $K_{(2)N}(x,y)$ from the unitary ensemble $\beta = 2$ (remembering the modifications associated to each value of $\beta = 1,4$), plus n corrections, in general (see (15) and (16)). Thus in the two non-χ ensembles we have $\rho_N^{(\beta)}(x) = \rho_N^{(2)}(x) + \rho_N^{(\beta)}{}_{\text{corr.}}(x)$, $\beta = 1,4$, where $\rho_N^{(2)}(x) = (N^{-1})K_{(2)N}(N^{-1}x, N^{-1}x)$ and $\rho_N^{(\beta)}{}_{\text{corr}}(x)$ constitute the 2 correction terms. When having calculated the matrix B and the four functions $\varepsilon\psi_i(x)$ both scaled corrections $\rho_N^{(\beta)}{}_{\text{corr.}}(x)$, $\beta = 1,4$, are given by the two last terms on the right hand side of (67) and (68). The contribution from the scaled spectral density $\rho_N^{(2)}(x)$ is an oscillating function in both ensembles $\beta = 1,4$, and the flat spectrum of $\rho_N^{(1)}(x)$ in the ensemble non-χGOE is therefore expected to be a result of the fact that $\rho_N^{(1)}{}_{\text{corr.}}(x)$ precisely cuts off the peaks of $\rho_N^{(2)}(x)$, and at the same time the term $\rho_N^{(4)}{}_{\text{corr.}}(x)$ must cause a highly oscillating $\rho_N^{(4)}(x)$. When plotting $\rho_N^{(\beta)}(x)$ together with the associated $\rho_N^{(2)}(x)$ we indeed observe these highly non-trivial features. The results are illustrated in figure 1 and 2.

In the limit $N \to \infty$ we have $\rho_N^{(\beta)}(x) \to \rho_s^{(\beta)}(x)$, $\beta = 1,4$. This convergence, however, is especially fast for x close to zero. This is illustrated in figure 3, where we have plotted $\rho_N^{(\beta)}(x)$, $\beta = 1,4$, for different values of N. The spectral density is an even function (see (40)) and for finite N we have the so-called half circles (with a hole in the vicinity of $x = 0$, due to the term x^{N_f} in the weight function), which

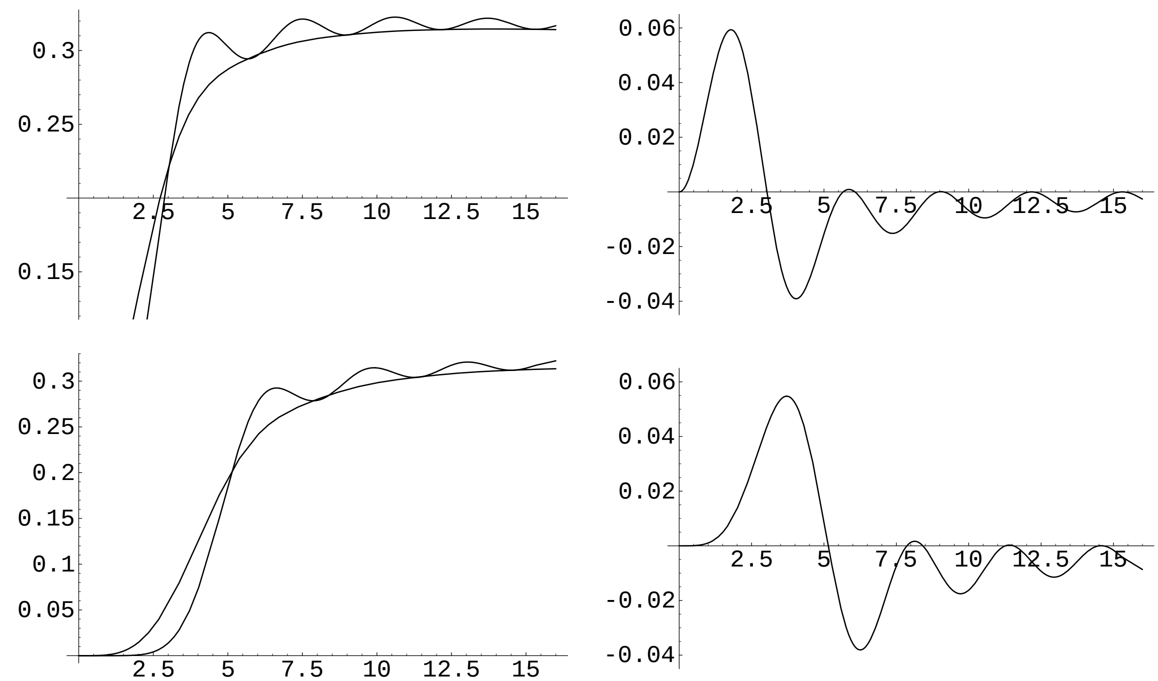

Figure 1: In non-χGOE we have $\rho_N^{(1)}(x) = \rho_N^{(2)}(x) + \rho_{N\ \mathrm{corr.}}^{(1)}(x)$, where $\rho_N^{(2)}(x)$ is from the associated unitary ensemble (given by orthonormal polynomials with respect to $\omega_{N_f}^{(1)}(x)^2$) and $\rho_{N\ \mathrm{corr.}}^{(1)}(x)$ is the scaled correction derived in last section. In left column the oscillating term $\rho_{40}^{(2)}(x)$ and the entire $\rho_{40}^{(1)}(x)$ are plotted together for $N = 40$ and the two values $N_f = 2$ (above) and $N_f = 4$ (under). The corresponding correction terms $\rho_{40\ \mathrm{corr.}}^{(1)}(x)$ are plotted in right column. We observe that the flat spectrum of non-χGOE is exactly a result of $\rho_{40\ \mathrm{corr.}}^{(1)}(x)$, cutting of the peaks of $\rho_{40}^{(2)}(x)$. In addition the curve of $\rho_{40}^{(1)}(x)$ moves away from zero when N_f is enlarged, due to the presence of the determinant function in the probability distribution (see (3)).

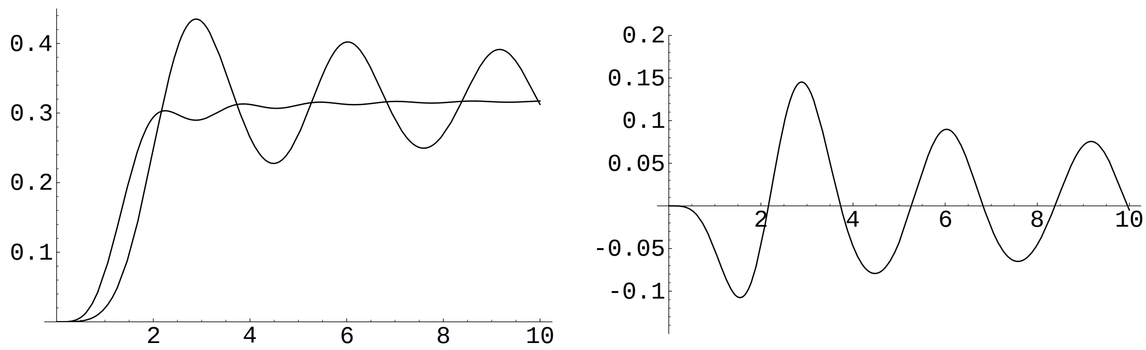

Figure 2: In non-χGSE we have $\rho_N^{(4)}(x) = \rho_N^{(2)}(x) + \rho_{N\,\mathrm{corr.}}^{(4)}(x)$, where $\rho_N^{(2)}(x)$ is from the associated unitary ensemble (given by orthonormal polynomials with respect to $\omega_{N_f}^{(4)}(x)$ and the replacement $N \to 2N$) and $\rho_{N\,\mathrm{corr.}}^{(4)}(x)$ is the scaled correction derived in last section. On the left the term $\rho_{40}^{(2)}(x)$ and the entire $\rho_{40}^{(1)}(x)$ are plotted together for $N = 40$ and the value $N_f = 4$. The corresponding correction term $\rho_{40\,\mathrm{corr.}}^{(4)}(x)$ is plotted on the right. We observe that $\rho_{40\,\mathrm{corr.}}^{(4)}(x)$ gives a highly oscillating $\rho_{40}^{(4)}(x)$ (a rigid spectrum).

is clearly seen in figure 3. Because of the the Gaussian distribution the microscopic spectral density fulfills $\rho_s(x) \to \pi^{-1}$ for $x \to \infty$ and thus the curves $\rho_N^{(\beta)}(x)$, $\beta = 1, 4$, are of order $\sim \pi^{-1}$ before the half circles start approaching zero (see also figure 1 and 2).

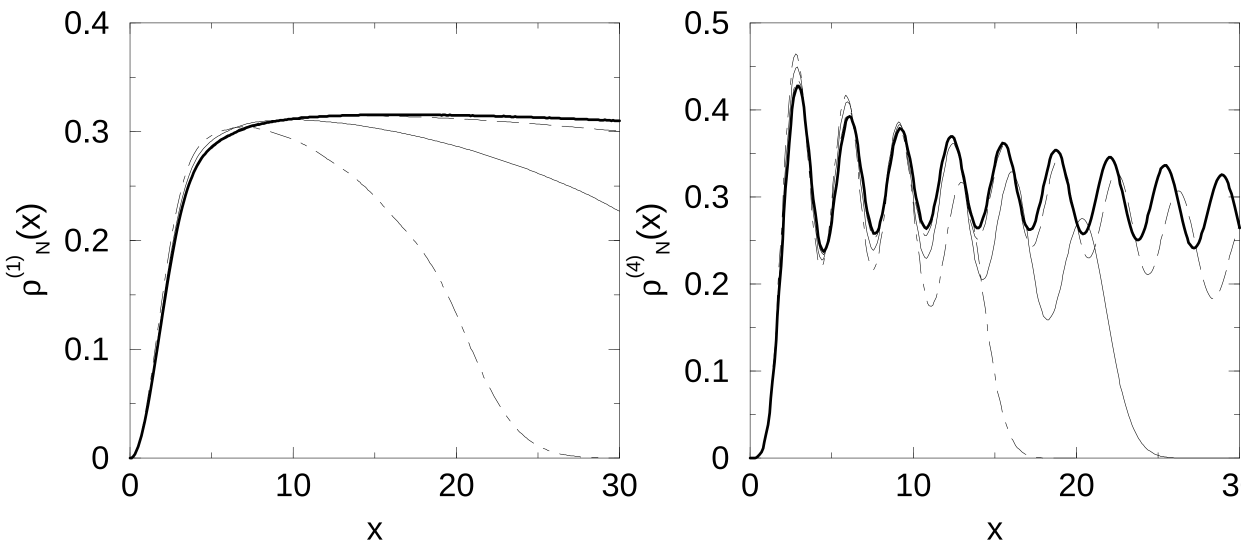

Figure 3: The scaled spectral density $\rho_N^{(\beta)}(x)$, for increasing N, on the left for $\beta = 1$ and on the right for $\beta = 4$. The convergens in the vicinity of $x = 0$ is clearly observed and we have $\rho_N^{(\beta)}(x) \approx \rho_s^{(\beta)}(x)$ near $z = 0$ as a consequence. In the limit $N \to \infty$ we have $\rho_N^{(\beta)}(x) \to \rho_s^{(\beta)}(x)$ and in the limit $x \to \infty$ we have $\rho_s^{(\beta)}(x) \to \pi^{-1}$.

5.1 The first spectral sum rule

The spectral sum rules derived in field theory are all given by the microscopic spectral density ρ_s of the Dirac operator [7, 27] and the result for ρ_s derived from RMT can therefore be verified through the knowledge of a spectral sum rule. The first sum involves the sum over the eigenvalues of the Dirac operator in second inverse power and in the limit $N \to \infty$ we have trivially

$$\langle \sum_{\lambda_k \neq 0} \frac{1}{(N\Sigma\lambda_k)^2} \rangle = \int_{-\infty}^{\infty} dz \, \frac{\rho_s(z)}{z^2}, \tag{99}$$

where the sum is over all $\lambda_k \neq 0$ (both positive and negative) and the average is taken over all gauge fields. For both field theories corresponding to the ensembles $\beta = 1, 4$ in $(2+1)$ dimensions (see table 1) the same spectral sum rule has been derived in [10, 11] and it reads

$$\langle \sum_{\lambda_k \neq 0} \frac{1}{(N\Sigma\lambda_k)^2} \rangle = \frac{4}{N_f}. \tag{100}$$

According to this, the first spectral sum rule involving the eigenvalues of the Dirac operator in two completely different field theories in $(2+1)$ dimensions should be identical.

With the help of our numerical expressions for $\rho_s^{(\beta)}(x)$ for the microscopic spectral density $\rho_s^{(\beta)}(x)$ in the two ensembles $\beta = 1, 4$ we thus have a consistency check on our results through (99) and (100). From figure 2 and 3 it is easily seen that the effect of the finite N is very small for a relatively large N. As seen our results does not agree with the spectral sum rule (100). In the ensemble $\beta = 4$ it

N_f	$\int_{-\infty}^{\infty} dz \, \frac{\rho_{40}^{(1)}(z)}{z^2}$	$\int_{-\infty}^{\infty} dz \, \frac{\rho_{40}^{(4)}(z)}{z^2}$	$\frac{4}{N_f}$
2	0.402	0.946	2
4	0.148	0.436	1
6	0.092	0.297	0.667
8	0.067	0.227	0.500
12	0.044	0.155	0.333

Table 4: The calculated first spectral sum rule, i.e the right hand side of (99), in the two ensembles $\beta = 1$ and $\beta = 4$ together with the spectral sum rule (100).

seems the spectral sum rule approaches $2/N_f$ for large N_f.
Next we turn to the most elementary check of our results, which in fact is an actual generation of matrices.

6 The Monte Carlo simulation

We have performed a Monte Carlo simulation in order to numerically verify the derived scaled spectral densities $\rho_N^{(\beta)}(x)$, $\beta = 1, 4$ The matrices in the generated ensemble are distributed according to (3), with all $m_f = 0$. The simulations are done for various values of the matrix size, N, and different values of N_f in the two cases $\beta = 1, 4$.
We diagonalize matrices from an ensemble made by a simple Metropolis algorithm. The eigenvalues,

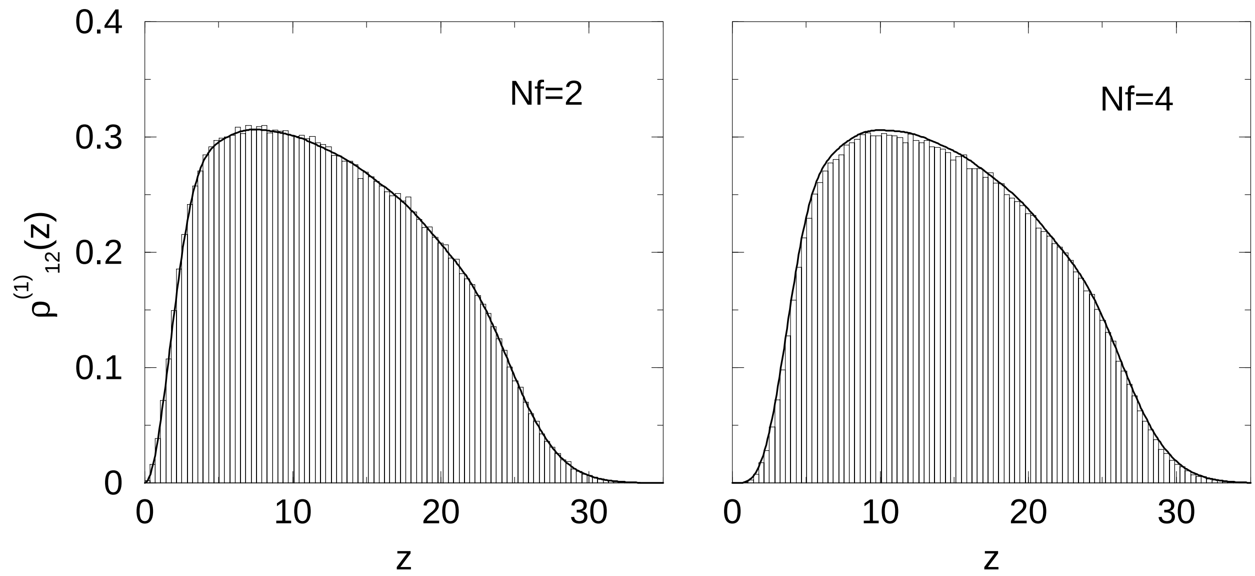

Figure 4: Scaled spectral density of the non-χGOE ensemble, compared with the Monte Carlo data, $N = 12$, $N_f = 2$ and 4.

x, are then collected to form a histogram, $h(x)$, which is rescaled

$$\begin{aligned} x &\rightarrow xN \equiv z, \\ h &\rightarrow \frac{h}{N} \end{aligned} \tag{101}$$

to obtain $\frac{1}{N}h(\frac{z}{N})$. The rescaled histogram is then compared to the plot of the analytically obtained scaled spectral density. Note that we compare the analytic plots for finite N with the corresponding simulation data. We thus expect the plots to match for the whole spectrum. For large N the scaled spectral density close to $z = 0$ has the form of the microscopic spectral density.
We handle the two cases $\beta = 1, 4$ individually as follows.

6.1 $\beta = 1$

The Metropolis step consist in changing the entries in the matrix and accepting these changes according to the action

$$S(M) = \frac{N}{4}\mathrm{Tr}M^2 - N_f \ln(\det M). \tag{102}$$

The changes is done in a way that explicitly maintain the symmetry of the matrix ($M = M^T$).
In this way we obtained an ensemble of 100,000 effectively uncorrelated matrices for $N = 12$ and 10,000 matrices for $N = 40$, both for $N_f = 2$ and 4. The resulting histograms together with their analytic equivalents can be seen in figure 4 and 5. We observe excellent agreement in all parts of the spectrum, for all values of the parameters N and N_f.

6.2 $\beta = 4$

In this case the Metropolis steps are accepted/rejected according to

$$S(M) = N\mathrm{Tr}M^2 - N_f \ln(\det M) \tag{103}$$

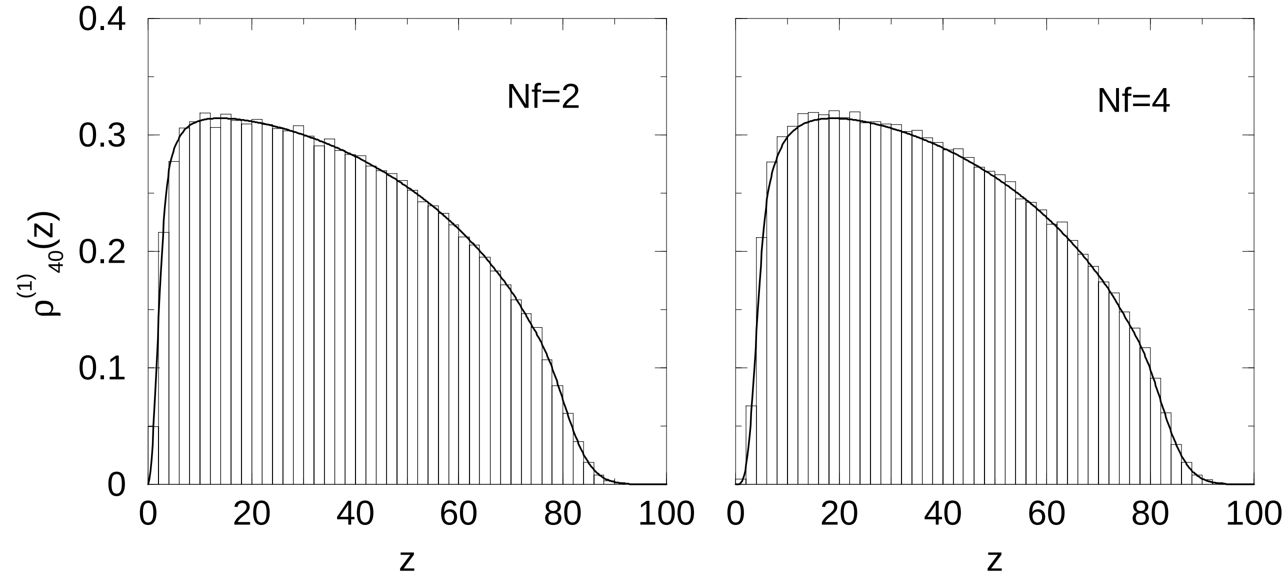

Figure 5: Scaled spectral density of the non-χGOE ensemble, compared with the Monte Carlo data, $N = 40$, $N_f = 2$ and 4.

A matrix from the symplectic ensemble can be represented by a $2N \times 2N$ matrix, M. This matrix can be written as a sum of direct-product matrices

$$M = M_0 e_0 + M_1 e_1 + M_2 e_2 + M_3 e_3 \tag{104}$$

where

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{105}$$

$$e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{106}$$

and the coefficients M_1, M_2, M_3 are real antisymmetric $N \times N$ matrices and M_0 is a real symmetric $N \times N$ matrix. In this way the matrix M has $2N$ eigenvalues which are doubly degenerate.
The Metropolis update works on the four real matrices, M_i, and M is then constructed from these and diagonalized. In this the way the symmetry and structure of the matrices is kept at all times. For the histogram only the N different eigenvalues is used.
When working in this $2N \times 2N$ representation the action, (103), has to be changed. The trace gets twice as large and the determinant is squared in this representation, so an overall factor of 1/2 should be included:

$$S(M) = \frac{N}{2} \mathrm{Tr} M^2 - \frac{N_f}{2} \ln(\det M) \tag{107}$$

These simulations are about twice as slow, so the maximum matrix size is half as big. We have done runs for $N = 8$ and 20, with $N_f = 2$ and 4, with the same statistics as in the $\beta = 1$ case. The plots are shown in figure 6 and 7. Here the agreement is even more striking. Zooming in on the first few eigenvalues in the $N = 20$ case we see absolutely perfect agreement indeed, see figure 8.

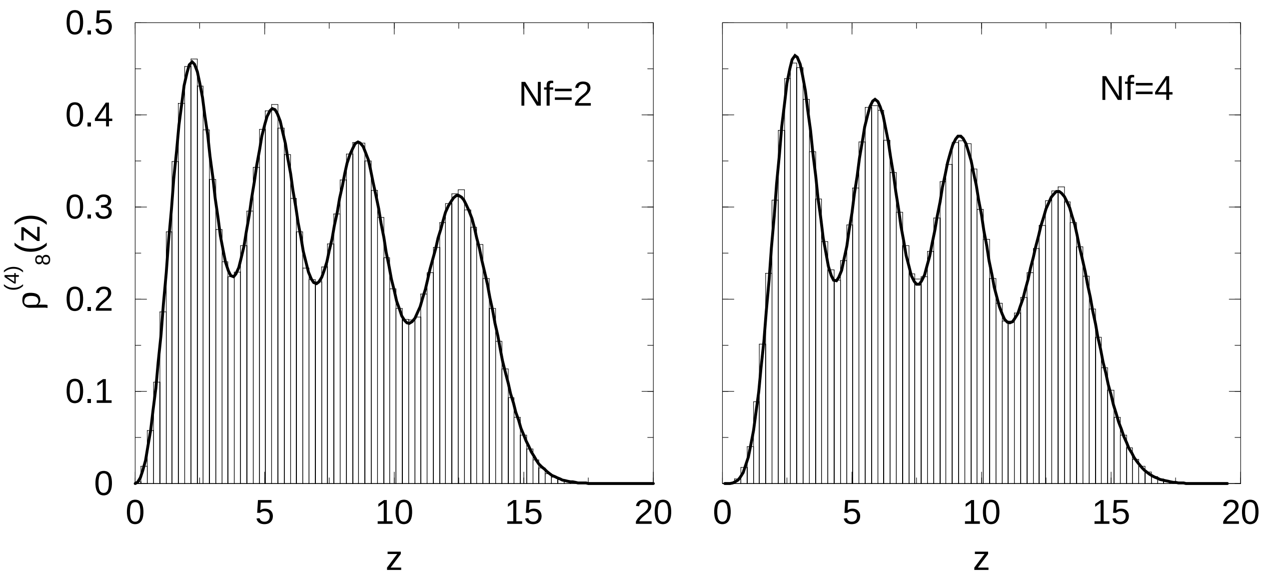

Figure 6: Scaled spectral density of the non-χGSE ensemble, compared with the Monte Carlo data, $N = 8$, $N_f = 2, 4$.

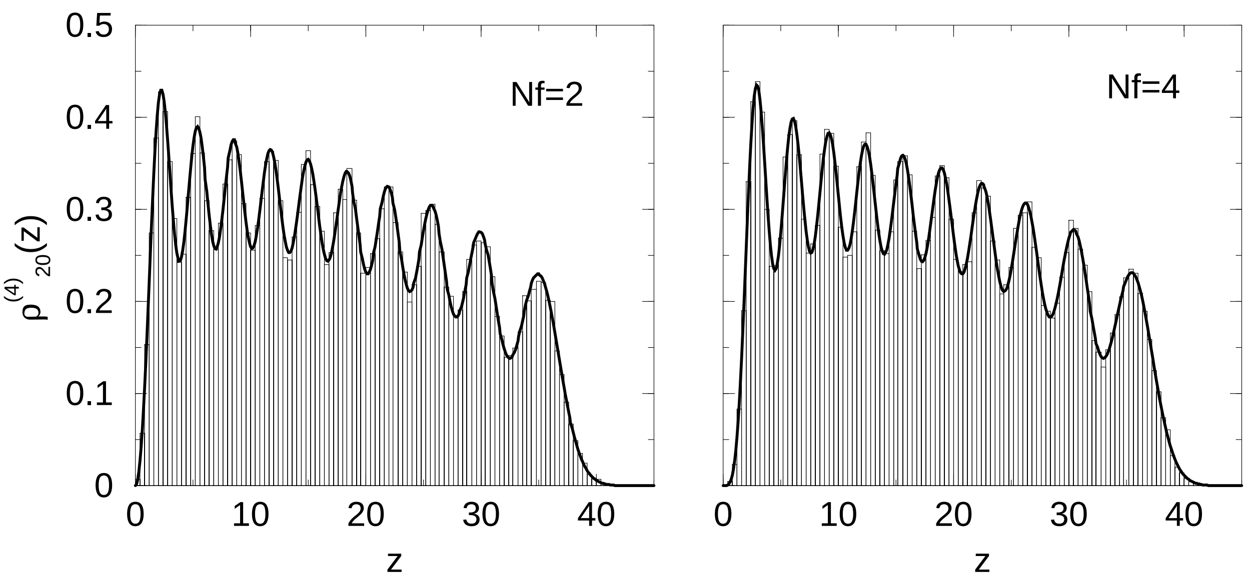

Figure 7: Scaled spectral density of the non-χGSE ensemble, compared with the Monte Carlo data, $N = 20$, $N_f = 2, 4$.

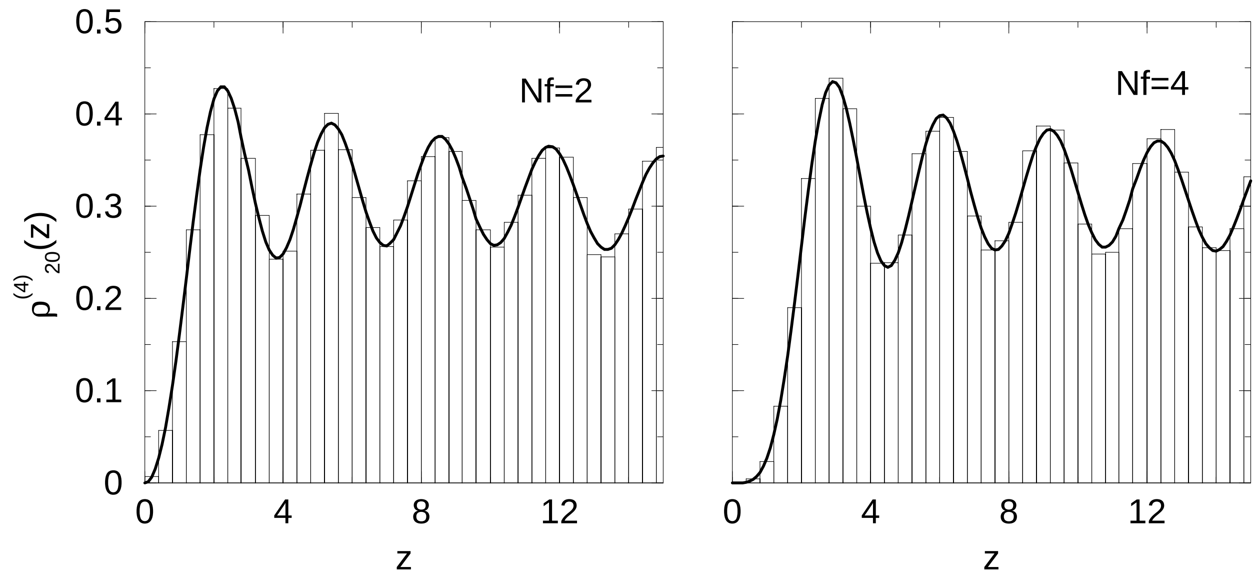

Figure 8: Same plot as figure 7 but zoomed in on small values of z.

7 Conclusions

The purpose of this study has been to derive the kernels $S_N^{(\beta)}(x,y)$, $\beta = 1,4$, from which all spectral correlation functions of the non-χGaussian orthogonal (non-χGOE) and symplectic (non-χGSE) ensembles can be determined. In the microscopic limit the two ensembles describes the microscopic spectral correlations of the low energy Dirac operator in two Yang-Mills theory in $(2+1)$ dimensions. Thus the universality class non-χGOE carries information about the spectral correlations in a $SU(2)$ gauge theory with fundamental fermions, while the universality class non-χGSE does the same for the $SU(N_c)$ gauge theory, with N_c arbitrary, and fermions in the adjoint representation.
With the help of Widom's new method we have derived the kernels $S_N^{(\beta)}(x,y)$, in non-χGOE ($\beta = 1$) and in non-χGSE ($\beta = 4$) for massless fermions. For non-χGOE our result are valid for all integers N_f and for non-χGSE we have a result for even N_f. We plotted the scaled spectral density $\rho_N^{(\beta)}(x) = N^{-1}S_N^{(\beta)}(N^{-1}x, N^{-1}x)$, for different values of the parameters N and N_f. The plots of $\rho_N^{(\beta)}(x)$ possesses the expected features, a flat spectrum for $\beta = 1$ and a highly oscillation spectrum for $\beta = 4$, and indeed we see perfect agreement with our computer simulated spectra. For large N the scaled spectral density $\rho_N^{(\beta)}(x)$ coincides with the microscopic spectral density $\rho_s(x)$ in the vicinity of $x = 0$. Our calculated spectral sum rules from $\rho_N^{(\beta)}(x)$ of course match the computer generated ones perfectly. We have not found agreement with the sum rules of [10, 11].

Recently the microscopic spectral densities of the massive non-χGOE and non-χGSE have been derived by a different method [29]. In the massless limit these result seem to match ours for large N.

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A Interchange of integration and the microscopic limit

In this Appendix we look at the large n-behavior of two types of general integrals involving generalized Laguerre polynomials. Specifically we examine when interchange of integration and the microscopic limit is allowed. To this end Lebesgue's Majorant Theorem is used. The first section is relevant for the calculation of $\varepsilon\psi_i$ in the limit $N \to \infty$, see (75). The result of section 2 is relevant for the calculation of the elements of the matrix B in the limit $N \to \infty$, see (74).

A.1 $\lim_{n\to\infty} \int_0^\infty dy\ n^{(1+a)-\alpha} L_n^\alpha(2y) y^a \exp(-y)$

We define

$$\mathcal{I}_n^{\alpha,a} \equiv \int_0^\infty dy\ n^{(1+a)-\alpha} L_n^\alpha(2y) y^a \exp(-y), \tag{108}$$

where n is a positive integer, $a > -1$ and α are real numbers. We wish to calculate

$$\lim_{n\to\infty} \mathcal{I}_n^{\alpha,a} = \lim_{n\to\infty} \int_0^\infty dy\ n^{(1+a)-\alpha} L_n^\alpha(2y) y^a \exp(-y), \tag{109}$$

and therefore examine when the interchange of the limit and the integration is allowed. In order to do this we employ Lebesgue's Majorant Theorem. At first we convince ourselfs that the sequence of functions in the integrand is convergent. The integral in (109) is rewritten by the substitution $u = (2ny)^{1/2}$, and we get

$$\lim_{n\to\infty} \mathcal{I}_n^{\alpha,a} = \lim_{n\to\infty} \int_0^\infty du\ n^{(1+a)-\alpha} L_n^\alpha(\frac{u^2}{n}) u^{2a+1} n^{-a-1} 2^{-a} \exp(-\frac{u^2}{2n}). \tag{110}$$

The integrand in (110)

$$f_n(u) \equiv n^{(1+a)-\alpha} L_n^\alpha(\frac{u^2}{n}) u^{2a+1} n^{-a-1} 2^{-a} \exp(-\frac{u^2}{2n}), \tag{111}$$

is convergent for $n \to \infty$: Inserting the well-known asymptotic formula [28]

$$\lim_{n\to\infty} \frac{1}{n^\alpha} L_n^\alpha(\frac{x^2}{n}) = x^{-\alpha} J_\alpha(2x), \tag{112}$$

we get

$$\lim_{n\to\infty} f_n(u) = \lim_{n\to\infty} n^{(1+a)-\alpha} L_n^\alpha(\frac{u^2}{n}) u^{2a+1} n^{-a-1} 2^{-a} \exp(-\frac{u^2}{2n}) = 2^{-a} u^{2a+1-\alpha} J_\alpha(2u). \tag{113}$$

Thus, the integrand in (109) is convergent. We must now look for an integrable majorant $\mathcal{M}(y)$, fulfilling for all n

$$|f_n(y)| = |n^{(1+a)-\alpha} L_n^\alpha(2y) y^a \exp(-y)| < \mathcal{M}(y). \tag{114}$$

If we can find such a function $\mathcal{M}(y)$, the Lebesgue Theorem states that interchange of the limit and integration is allowed in (109). In this case we can insert (113) in (109).

We start by splitting up the integral into integrals, in the first integrating over the interval $]0, \epsilon[$ for some finite ϵ and in the second over $]\epsilon, \infty[$. Since the first integral is finite for every n we focus only on the second integral. By choosing a sufficiently large n' we can use an asymptotic expression for the integrand. We then check if the absolute value of this expression has an integrable majorant for all $n > n'$, see (114). Using the asymptotic property (valid for $x > 0$) [25]

$$L_n^\alpha(x) = \frac{1}{\pi} e^{\frac{x}{2}} x^{-(\frac{2\alpha-1}{4})} n^{(\frac{2\alpha-1}{4})} \cos[2(nx)^{\frac{1}{2}} - \frac{\alpha\pi}{2} - \frac{\pi}{4}] + \mathcal{O}(n^{\frac{2\alpha-3}{4}}). \tag{115}$$

for $L_n^\alpha(2y)$ in (109), we get

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{I}_n^{\alpha,a} \\
&\approx \lim_{n\to\infty}\int_\epsilon^\infty dy\ n^{(1+a-\alpha)} \\
&\quad \times\Big\{\pi^{-1/2}\exp(y)y^{-(\frac{2\alpha+1}{4})}2^{-(\frac{2\alpha+1}{4})}n^{(\frac{2\alpha-1}{4})}\cos\Big[2(2ny)^{\frac{1}{2}}-\frac{\alpha\pi}{2}-\frac{\pi}{4}\Big]\Big\}y^a\exp(-y) \\
&= \pi^{-1/2}2^{-(\frac{2\alpha+1}{4})}\lim_{n\to\infty} n^{(1+a-\frac{\alpha}{2}-\frac{1}{4})}\int_\epsilon^\infty dy\ y^{-\frac{2\alpha+1}{4}+a}\cos\Big[2(2ny)^{\frac{1}{2}}-\frac{\alpha\pi}{2}-\frac{\pi}{4}\Big],
\end{aligned} \tag{116}$$

for $n > n'$.
We know that the sequence of functions in (116) is convergent, and the function

$$\mathcal{M}(y) = c\,\pi^{-1/2}2^{-(\frac{2\alpha+1}{4})}\ y^{-\frac{2\alpha+1}{4}+a}, \qquad c > 1, \tag{117}$$

is then a majorant to the absolute value of the integrand in (109) for $n > n'$ and some $c > 1$, which take account of the term $\mathcal{O}$ in (115). Since we need $\int_\epsilon^\infty dy\,\mathcal{M}(y) < \infty$ we must have $-(\alpha/2+1/4)+a < -1$. So we conclude, *if*

$$\alpha/2 - a > 3/4, \tag{118}$$

then (117) *is a integrable majorant and we can then interchange the order of the limit and the integral in (109).* Asumming this in (113), gives us

$$\lim_{n\to\infty}\mathcal{I}_n^{\alpha,a} = 2^{-a}\int_0^\infty du\ u^{2a+1-\alpha}J_\alpha(2u), \tag{119}$$

for $\alpha/2 - a > 3/4$.
On the contrary, if $\alpha/2 - a \le 3/4$, then the integrand in (109) has no majorant, $\mathcal{M}(y)$, which fulfills $\int_0^\infty dy\mathcal{M}(y) < \infty$. This follows from the fact that the integral from ϵ to infinite of the absolut value of the integrand in (116) is not finite. Substituting $z = 2ny$ in (116) gives an integrand on the form $z^{-\delta}\cos(2z^{1/2}+k)$, where $\delta = -(\alpha/2+1/4)+a$ (the n dependence vanishes of course). The smallest majorant to $|z^{-\delta}\cos(2z^{1/2}+k)|$, is $C|z^{-\delta}\cos(2z^{1/2}+k)|$, for some $C > 1$. This majorant is not integrable when $\delta = -(\alpha/2+1/4)+a \ge -1$, and thus we have no integrable majorant. When the absolut value of the asymptotic function does not integrate to a finite number, then of course no majorant exist for which the integral is finite (see (114)).
We conclude, when

$$\alpha/2 - a \le 3/4, \tag{120}$$

interchange of integration and the limit in (109) *is not legal.* In this case we first have to solve the integral in (109) and subsequently derive an expression in the limit $n \to \infty$.

A.2 $\lim_{n\to\infty}\int_0^\infty dt\ n^\lambda L_n^\alpha(2t)t^\beta\exp(-t)\int_0^t du\ L_n^{\bar\alpha}(2u)u^{\bar\beta}\exp(-u)$

We will now look at the limit $n \to \infty$ of integrals of the following type

$$\mathcal{I}_n^{\alpha,\beta,\bar\alpha,\bar\beta} \equiv n^\lambda\int_0^\infty dt\ L_n^\alpha(2t)t^\beta\exp(-t)\int_0^t du\ L_n^{\bar\alpha}(2u)u^{\bar\beta}\exp(-u), \tag{121}$$

where $\alpha, \bar\alpha, \beta, \bar\beta > -1$ and λ is real and n is a positive integer. The matrix B in section 5 is given by integrals of this type, and we are interested in an expression for B in the limit $n \to \infty$. It is assumed that the integrand is convergent for $n \to \infty$, meaning that λ has a certain value, the size of which is

irrelevant for the question addressed in this Appendix.
Analogous to the previous section we investigate whether the limit and the integration can be interchanged in the expression

$$\lim_{n\to\infty} \mathcal{I}_n^{\alpha,\beta,\bar{\alpha},\bar{\beta}} = \lim_{n\to\infty} \int_0^\infty dt\ n^\lambda L_n^\alpha(2t) t^\beta \exp(-t) \int_0^t du\ L_n^{\bar{\alpha}}(2u) u^{\bar{\beta}} \exp(-u). \tag{122}$$

By assumtion the sequence

$$f_n(t) \equiv n^\lambda L_n^\alpha(2t) t^\beta \exp(-t) \int_0^t du\ L_n^{\bar{\alpha}}(2u) u^{\bar{\beta}} \exp(-u), \tag{123}$$

is convergent (compare with (111) og (113)). We must determine whether or not the absolut value of the integrandt in (122), that is $|f_n|$, has an integrabel majorant $\mathcal{M}(t)$. Like in Appendix B.1 we persuit this question by choosing a large n and put in the asymptotic relation (115) for both Laguerre polynomials :

$$\lim_{n\to\infty} \mathcal{I}_n^{\alpha,\beta,\bar{\alpha},\bar{\beta}} \approx \pi^{-1} 2^{-\frac{\alpha+\bar{\alpha}+1}{2}} \lim_{n\to\infty} n^\lambda \quad \times$$
$$\int_\epsilon^\infty dt\ |\ n^{\frac{\alpha}{2}-\frac{1}{4}}\ t^{-(\frac{\alpha}{2}+\frac{1}{4}-\beta)} \cos\left[2(2nt)^{\frac{1}{2}} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] \int_\epsilon^t du\ n^{\frac{\bar{\alpha}}{2}-\frac{1}{4}} u^{-(\frac{\bar{\alpha}}{2}+\frac{1}{4}-\bar{\beta})} \cos\left[2(2nu)^{\frac{1}{2}} - \frac{\bar{\alpha}\pi}{2} - \frac{\pi}{4}\right]\ |. \tag{124}$$

Like in Appendix B.1 we skip all contributions involving irrelevant integrals over $]0,\epsilon[$, because in these cases the exact integrals are finite. The sequence in (122), that is f_n above, is convergent by assumtion. Thus for all n the absolut value of the integrand in (124) is always smaller than the function

$$c\ t^{-(\frac{\alpha}{2}+\frac{1}{4}-\beta)} \int_\epsilon^t du\ u^{-(\frac{\bar{\alpha}}{2}+\frac{1}{4}-\bar{\beta})}, \tag{125}$$

where $c > 1$. Letting $\delta = -(\frac{\alpha}{2} + \frac{1}{4} - \beta)$ and $\rho = -(\frac{\bar{\alpha}}{2} + \frac{1}{4} - \bar{\beta})$ we have trivially

$$c \int_\epsilon^\infty dt\ t^\delta \int_\epsilon^t du\ u^\rho = \frac{c}{\rho+1} \int_\epsilon^\infty dt\ \left(t^{\rho+\delta+1} - t^\delta \epsilon^{\rho+1}\right). \tag{126}$$

We see immediately that for

$$\rho + \delta + 1 = -\frac{\alpha + \bar{\alpha} - 1}{2} + \beta + \bar{\beta} < -1 \quad \text{and} \quad \delta = -(\frac{\alpha}{2} + \frac{1}{4} - \beta) < -1 \tag{127}$$

the integral in (126) is finite. For large n we therefore have that the absolute value of the integrand in (123) always is smaller than the function (125), and if (127) is valid the function (125) is an integrable majorant.
On the contrary we see from (124) in case of $\delta = -(\frac{\alpha}{2} + \frac{1}{4} - \beta) \geq -1$ and $\rho = -(\frac{\bar{\alpha}}{2} + \frac{1}{4} - \bar{\beta}) \geq -1$, then the integral over t is *divergent*. It follows that the absolut value of the integrand in (122) does not have an integrable majorant, since the asymptotic expression does not even have one.
The other cases of values of ρ and δ are not relevant for us and we ignore them.

Summarizing the discussion we have

(1) If (127) is valid, then the absolut value of the integrand in (122) has an integrable majorant (125). Thus interchange of the limit and integration in (122) is legal:

$$\lim_{n\to\infty} \mathcal{I}_n^{\alpha,\beta,\bar{\alpha},\bar{\beta}} = \int_0^\infty dt\ \lim_{n\to\infty} n^\lambda L_n^\alpha(2t) t^\beta \exp(-t) \int_0^t du\ L_n^{\bar{\alpha}}(2u) u^{\bar{\beta}} \exp(-u). \tag{128}$$

(2) If $-(\frac{\alpha}{2}+\frac{1}{4}-\beta)\geq -1$ *and* $-(\frac{\bar{\alpha}}{2}+\frac{1}{4}-\bar{\beta})\geq -1$, then the absolut value of the integrand in (122) has no majorant $\mathcal{M}(t)$ fulfilling $\int_0^\infty dt\ \mathcal{M}(t)<\infty$. Therefore interchange of the limit and the integration in (122) is not allowed. In this case we must solve the integral before the limit is taken.

B Two integrals solved

In this Appendix we find expressions for two different types of integrals. We wish to derive an expression for the function

$$\mathcal{E}_{[\bar{\alpha},\bar{\beta},n]}(x)\equiv \varepsilon\ L_n^{\bar{\alpha}}(x^2)x^{\bar{\beta}}\exp(-\frac{x^2}{2})=\int_{-\infty}^{\infty}dy\ \varepsilon(x-y)L_n^{\bar{\alpha}}(y^2)y^{\bar{\beta}}\exp(-\frac{y^2}{2}), \tag{129}$$

where $\bar{\alpha}>-1$ is a real number and n, $\bar{\beta}$ are integers.

For integers m and n we calculate the following numbers

$$\begin{aligned}\mathcal{B}_{ij}&\equiv&(\varepsilon\ L_n^{\bar{\alpha}}(x^2)x^{\bar{\beta}}\exp(-\frac{x^2}{2})\ ,\ L_m^{\alpha}(x^2)x^{\beta}e^{-\frac{x^2}{2}})=\int_{-\infty}^{\infty}dx\ L_m^{\alpha}(x^2)x^{\beta}e^{-\frac{x^2}{2}}\varepsilon\ L_n^{\bar{\alpha}}(x^2)x^{\bar{\beta}}e^{-\frac{y^2}{2}}\\&=&\int_{-\infty}^{\infty}dx\ L_m^{\alpha}(x^2)x^{\beta}e^{-\frac{x^2}{2}}\int_{-\infty}^{\infty}dy\ \varepsilon(x-y)L_n^{\bar{\alpha}}(y^2)y^{\bar{\beta}}e^{-\frac{y^2}{2}},\end{aligned} \tag{130}$$

The index i refers to m,α, and β, while j refers to $n,\bar{\alpha}$ and $\bar{\beta}$. m, n, β and $\bar{\beta}$ are integers and α and $\bar{\alpha}$ are real numbers greater than -1.
The function (129) is a part of the inner product in $\mathcal{B}_{ij}$ and first we therefore derive an expression for (129).

B.1 $\varepsilon\ L_n^{\bar{\alpha}}(x^2)x^{\bar{\beta}}\exp(-\frac{x^2}{2})$

We start by defining

$$E(\lambda,x)\equiv \varepsilon x^{\lambda}\Big(xe^{-\frac{x^2}{2}}\Big), \tag{131}$$

where $\lambda\geq -1$ is an integer. We have the following recursion formula

$$\begin{aligned}E(\lambda,x)&=&\int_{-\infty}^{\infty}dy\ \varepsilon(x-y)y^{\lambda}\Big(ye^{-\frac{y^2}{2}}\Big)\\&=&\frac{1}{2}\Big[\int_{-\infty}^{x}dy-\int_{x}^{\infty}dy\Big]y^{\lambda}\Big(ye^{-\frac{y^2}{2}}\Big)\\&=&-[\frac{1}{2}e^{-\frac{y^2}{2}}y^{\lambda}]_{-\infty}^{x}+[\frac{1}{2}e^{-\frac{y^2}{2}}y^{\lambda}]_{x}^{\infty}+\int_{-\infty}^{\infty}dy\ \varepsilon(x-y)\lambda y^{\lambda-1}\Big(e^{-\frac{y^2}{2}}\Big)\\&=&-e^{-\frac{x^2}{2}}x^{\lambda}+\lambda\int_{-\infty}^{\infty}dy\ \varepsilon(x-y)\ y^{\lambda-2}\Big(ye^{-\frac{y^2}{2}}\Big)\\&=&-e^{-\frac{x^2}{2}}x^{\lambda}+\lambda E(\lambda-2,x),\end{aligned} \tag{132}$$

for $\lambda\geq 1$. For $\lambda=-1$ we have

$$\begin{aligned}E(-1,x)&=&\int_{-\infty}^{\infty}dy\ \varepsilon(x-y)e^{-\frac{y^2}{2}}\\&=&\frac{1}{2}\Big\{\int_{-\infty}^{-x}dy+\int_{-x}^{x}dy-\int_{x}^{\infty}\Big\}\ e^{-\frac{y^2}{2}}\\&=&\int_0^x dy\ e^{-\frac{x^2}{2}}=\sqrt{\frac{\pi}{2}}\mathrm{erf}(\frac{x}{\sqrt{2}}).\end{aligned} \tag{133}$$

And for $\lambda = 0$ we get

$$E(0,x) \quad = \quad \int_{-\infty}^{\infty} dy \; \varepsilon(x-y) y e^{-\frac{y^2}{2}} = -e^{-\frac{x^2}{2}}. \tag{134}$$

For λ *even* we immediately get from (132) and (134)

$$E(\lambda,x) = -e^{-\frac{x^2}{2}} \Big(x^\lambda + \lambda x^{\lambda-2} + \ldots + \lambda(\lambda-2)\cdots 4\cdot 2 \Big). \tag{135}$$

In the case of *odd* λ, (132) and (133) give us :

$$E(\lambda,x) = -e^{-\frac{x^2}{2}} \Big(x^\lambda + \lambda x^{\lambda-2} + \ldots + \lambda(\lambda-2)\cdots 5\cdot 3x \Big) + \lambda(\lambda-2)\cdots 5\cdot 3 \sqrt{\frac{\pi}{2}} \mathrm{erf}(\frac{x}{\sqrt{2}}). \tag{136}$$

Writing out the terms of the Laguerre polynomials through [25]

$$L_n^\alpha(x) = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{x^m}{m!} = \sum_{m=0}^{n} (-1)^m \frac{\Gamma(n+\alpha+1)}{\Gamma(n-m+1)\Gamma(m+\alpha+1)} \frac{x^m}{m!}, \tag{137}$$

we get

$$\begin{aligned}
\varepsilon \; L_n^{\bar\alpha}(x^2) x^{\bar\beta} \exp(-\frac{x^2}{2}) \quad &= \quad \int_{-\infty}^{\infty} dy \; \varepsilon(x-y) L_n^{\bar\alpha}(y^2) y^{\bar\beta} e^{-\frac{y^2}{2}} \\
&= \quad \sum_{i=0}^{n} a_i \int_{-\infty}^{\infty} dy \; \varepsilon(x-y) \; y^{2i+\bar\beta-1} \Big(y e^{-\frac{y^2}{2}} \Big) \\
&= \quad a_0 \int_{-\infty}^{\infty} dy \; \varepsilon(x-y) \; y^{\bar\beta-1} \Big(y e^{-\frac{y^2}{2}} \Big) + a_1 \int_{-\infty}^{\infty} dy \; \varepsilon(x-y) \; y^{\bar\beta+1} \Big(y e^{-\frac{y^2}{2}} \Big) + \\
&\qquad \ldots + a_n \int_{-\infty}^{\infty} dy \; \varepsilon(x-y) \; y^{2n+\bar\beta-1} \Big(y e^{-\frac{y^2}{2}} \Big),
\end{aligned} \tag{138}$$

where a_i are the coefficients to x^i in (137).
Assuming that $\bar\beta$ is *odd*, all powers in (138) are even and we use (135) on each term to give us

$$\begin{aligned}
&\varepsilon \; L_n^{\bar\alpha}(x^2) x^{\bar\beta} \exp(-\frac{x^2}{2}) \\
&= \quad -e^{-\frac{x^2}{2}} \Big[a_0 \Big(x^{\bar\beta-1} + (\bar\beta-1) x^{\bar\beta-3} + \ldots + (\bar\beta-1)(\bar\beta-3)\cdots 4\cdot 2 \Big) + \\
&\qquad a_1 \Big(x^{\bar\beta+1} + (\bar\beta+1) x^{\bar\beta-1} + \ldots + (\bar\beta+1)(\bar\beta-1)\cdots 4\cdot 2 \Big) + \cdots\cdots \\
&\qquad + a_n \Big(x^{\bar\beta+2n-1} + (\bar\beta+2n-1) x^{\bar\beta-3} + \ldots + (\bar\beta+2n-1)(\bar\beta+2n-3)\cdots 4\cdot 2 \Big) \Big] \\
&= \quad -e^{-\frac{x^2}{2}} \Big[\Big\{ a_0 + (\bar\beta+1)a_1 + (\bar\beta+3)(\bar\beta+1)a_2 + \ldots + (\bar\beta+2n-1)(\bar\beta+2n-3)\cdots(\bar\beta+1)a_n \Big\} \times \\
&\qquad \Big(x^{\bar\beta-1} + (\bar\beta-1)x^{\bar\beta-3} + (\bar\beta-1)(\bar\beta-3)x^{\bar\beta-5} + \cdots + (\bar\beta-1)(\bar\beta-3)\cdots 4\cdot 2 \Big) \\
&\qquad + x^{\bar\beta+1} \Big(a_1 + (\bar\beta+3)a_2 + (\bar\beta+5)(\bar\beta+3)a_3 + \ldots + (\bar\beta+2n-1)(\bar\beta+2n-3)\cdots(\bar\beta+3)a_n \Big) + \\
&\qquad x^{\bar\beta+3} \Big(a_2 + (\bar\beta+5)a_3 + \ldots + (\bar\beta+2n-1)\cdots(\bar\beta+5)a_n \Big) + \ldots \\
&\qquad\qquad \ldots + x^{\bar\beta+2n-1} a_n \Big].
\end{aligned} \tag{139}$$

When $\bar\beta$ is *even* we use (136) in (138), and we get

$$\begin{aligned}
&\varepsilon\ L_n^{\bar\alpha}(x^2)x^{\bar\beta}\exp(-\frac{x^2}{2})\\
&= \ -e^{-\frac{x^2}{2}}\Big[a_0\Big(x^{\bar\beta-1}+(\bar\beta-1)x^{\bar\beta-3}+\ldots+(\bar\beta-1)(\bar\beta-3)\cdots5\cdot3\ \mathrm{erf}(\frac{x}{\sqrt2})\Big)+\\
&\qquad a_1\Big(x^{\bar\beta+1}+(\bar\beta+1)x^{\bar\beta-1}+\ldots+(\bar\beta+1)(\bar\beta-1)\cdots5\cdot3\ \mathrm{erf}(\frac{x}{\sqrt2}\Big)+\cdots\cdots\\
&\quad +a_n\Big(x^{\bar\beta+2n-1}+(\bar\beta+2n-1)x^{\bar\beta-3}+\ldots+(\bar\beta+2n-1)(\bar\beta+2n-3)\cdots5\cdot3\ \mathrm{erf}(\frac{x}{\sqrt2}\Big)\Big].
\end{aligned}\tag{140}$$

Using the definitions [28]

$$(2k)!!\equiv 2k(2k-2)\cdots4\cdot\ 2=2^k\ \Gamma(k+1),\tag{141}$$

$$(2k-1)!!\equiv(2k-1)(2k-3)\cdots3\cdot\ 1=\pi^{-\frac12}\ 2^k\ \Gamma(k+\frac12),\tag{142}$$

where k is a positive integer, we can write (139) and (140) in a more compact way.
For every a_i, $0\le i\le n$, the righthand side of eq. (139) is equal to $-e^{-x^2/2}$ times a polynomial of order $(\bar\beta+2i-1)$. This polynomial can be rewritten as

$$a_i\sum_{j=0}^{\frac{\bar\beta-1}{2}+i}x^{\bar\beta-1-2j+2i}\ 2^j\ \frac{\Gamma(\frac{\bar\beta+1}{2}+i)}{\Gamma(\frac{\bar\beta+1}{2}+i-j)}.\tag{143}$$

A similar expression exist for the polynomial in the framed part of (140). Using these expressions for the polynomial parts of (140) and (139) and inserting the Laguerre coefficients a_i of x^i in (137) in (140) and (139) leads to

$$\boxed{\begin{aligned}
&\mathcal{E}_{[\bar\alpha,\bar\beta,n]}(x)=\varepsilon\ L_n^{\bar\alpha}(x^2)x^{\bar\beta}\exp(-\frac{x^2}{2})=\\
&-e^{\frac{-x^2}{2}}\Big[\sum_{i=0}^{n}\frac{(-1)^i}{\Gamma(i+1)}\frac{\Gamma(n+1+\bar\alpha)}{\Gamma(n+1-i)\Gamma(\bar\alpha+1+i)}\sum_{j=0}^{\frac{\bar\beta-1}{2}+i\ \{-\frac32\}}x^{(\bar\beta-1-2j+2i)}\ 2^j\frac{\Gamma(\frac{\bar\beta+1}{2}+i)}{\Gamma(\frac{\bar\beta+1}{2}+i-j)}\Big]\\
&\Big\{+\sum_{i=0}^{n}\frac{(-1)^i}{\Gamma(i+1)}\frac{\Gamma(n+1+\bar\alpha)}{\Gamma(n+1-i)\Gamma(\bar\alpha+1+i)}2^{(i+\frac{\bar\beta}{2})}\Gamma\Big(i+\frac{\bar\beta+1}{2}\Big)\Big[-e^{-\frac{x^2}{2}}\frac{x}{\sqrt\pi}\ +\sqrt{\frac12}\ \mathrm{erf}(\frac{x}{\sqrt2})\Big]\Big\}.
\end{aligned}}\tag{144}$$

For *even* $\bar\beta$ we must include the terms in the brackets $\{\ldots\}$, while these are neglected for *odd* $\bar\beta$. For $\bar\beta=0$ a modification of the solution with the brackets is valid : for $i=0$ we must delete the term $-xe^{-x^2/2}/\sqrt\pi$. All contributions from a negative upper limit in the summation are set to zero (for instance for $\bar\beta=2$ and $i=0$ the term involving $\sum_{j=0}^{-1}$ equals zero).

B.2 $\int_{-\infty}^{\infty} dx\ L_m^{\alpha}(x^2)x^{\beta}e^{-\frac{x^2}{2}}\varepsilon\ L_n^{\bar{\alpha}}(x^2)x^{\bar{\beta}}e^{-\frac{x^2}{2}}$

Using (144) we are now able to calculate (130). Since β, $\bar{\beta}$ are integers, the interior and exterior functions in (130)

$$f_{int}(x) \equiv L_n^{\bar{\alpha}}(x^2)x^{\bar{\beta}}e^{-\frac{x^2}{2}}$$

and

$$f_{ext}(x) \equiv L_m^{\alpha}(x^2)x^{\beta}e^{-\frac{x^2}{2}}$$

are either even or odd. In this case it is easily shown that $\mathcal{B}_{ij}$ is non-zero only when $\bar{\beta}$ is even and β is odd or vice visa. In the following we assumed that β and $\bar{\beta}$ fulfills this. Using the relation $\mathcal{B}_{ij} = -\mathcal{B}_{ji}$ we can make sure that the function in the interior integral in (130) is *odd*. If $\bar{\beta}$ is *odd* this is automatically the case and we have

$$\mathcal{B}_{ij} = \int_{-\infty}^{\infty} dx\ f_{ext}(x)\varepsilon f_{int}(x), \tag{145}$$

with $\varepsilon f_{int}(y)$ given by (144). In case of $\bar{\beta}$ *even* we simply use $\mathcal{B}_{ij} = -\mathcal{B}_{ji}$ to switch the two functions

$$\mathcal{B}_{ij} = -\mathcal{B}_{ji} = -\int_{-\infty}^{\infty} dx\ f_{int}(x)\varepsilon f_{ext}(x). \tag{146}$$

where now $\varepsilon f_{ext}(x)$ is given by (144).
So from now on we assume $\bar{\beta}$ is *odd* and β *even*.
Denoting by $p_{\bar{\beta}-1+2n}$ the $(\bar{\beta}-1+2n)$ order polynomial from (144), we want to calculate

$$\mathcal{B}_{ij} \quad = \quad \int_{-\infty}^{\infty} dx\ L_m^{\alpha}(x^2)x^{\beta}e^{-x^2}p_{\bar{\beta}-1+2n}(x) \tag{147}$$

Since both $p_{\bar{\beta}-1+2n}$ and $L_m^{\alpha}(x^2)x^{\beta}e^{-\frac{x^2}{2}}$ are *even*, we have

$$\mathcal{B}_{ij} \quad = \quad -2\int_{0}^{\infty} dx\ L_m^{\alpha}(x^2)x^{\beta}e^{-x^2}p_{\frac{\bar{\beta}-1+2n}{2}}(x^2). \tag{148}$$

Making the substitution $z = x^2$, gives

$$\mathcal{B}_{ij} \quad = \quad -\int_{0}^{\infty} dz\ L_m^{\alpha}(z)z^{\frac{\beta-1}{2}}e^{-z}p_{\bar{\beta}-1+2n}(z^{\frac{1}{2}}). \tag{149}$$

Let us rewrite $p_{\bar{\beta}-1+2n}$ using (139)

$$\begin{aligned} p_{\bar{\beta}-1+2n}(z^{\frac{1}{2}}) \quad = \quad & b_{\bar{\beta}+2n-1}z^{\frac{\bar{\beta}+2n-1}{2}} + b_{(\bar{\beta}+2n-3)}\ z^{\frac{\bar{\beta}+2n-3}{2}} + \ldots + b_{(\bar{\beta}+3)}\ z^{\frac{\bar{\beta}+3}{2}} + b_{(\bar{\beta}+1)}\ z^{\frac{\bar{\beta}+1}{2}} + \\ & T\Big(z^{\frac{\bar{\beta}-1}{2}} + b_{(\bar{\beta}-3)}z^{\frac{\bar{\beta}-3}{2}} + b_{(\bar{\beta}-5)}z^{\frac{\bar{\beta}-5}{2}} + \ldots + b_0\Big) \end{aligned} \tag{150}$$

The coefficients b_i are given by products of $\bar{\beta}$ and a_i, and T is the factor in brackets ($\{\ldots\}$) in (139). Using (141) and (142) T can be reduced to :

$$\begin{aligned} T \quad & = \quad \Big\{a_0 + (\bar{\beta}+1)a_1 + (\bar{\beta}+3)(\bar{\beta}+1)a_2 + \ldots + (\bar{\beta}+2n-1)(\bar{\beta}+2n-3)\cdots(\bar{\beta}+1)a_n\Big\} \\ & = \quad \sum_{i=0}^{n}(-1)^i\ 2^i\frac{\Gamma(\frac{\bar{\beta}+1}{2}+i)}{\Gamma(\frac{\bar{\beta}+1}{2})\Gamma(i+1)}\frac{\Gamma(n+1+\bar{\alpha})}{\Gamma(n+1-i)\Gamma(\bar{\alpha}+1+i)}. \end{aligned} \tag{151}$$

By insertion of (150) in (149) we get

$$\begin{aligned}\mathcal{B}_{ij} &= -\int_0^\infty dz\ L_m^\alpha(z)e^{-z}z^{\frac{\bar\beta+\beta}{2}}\Big[b_{(\bar\beta+2n-1)}z^{n-1}+b_{(\bar\beta+2n-3)}z^{n-2}+\ldots+b_{(\bar\beta+3)}z+b_{(\bar\beta+1)}\Big]\\ &\qquad -T\int_0^\infty dz\ L_m^\alpha(z)e^{-z}z^{\frac{\bar\beta-1}{2}}\Big(z^{\frac{(\bar\beta-1)}{2}}+b_{(\bar\beta-3)}z^{\frac{\bar\beta-3}{2}}+b_{(\bar\beta-5)}z^{\frac{\bar\beta-5}{2}}+\ldots+b_0\Big). \quad (152)\end{aligned}$$

We restrict the parameters to the following four cases

$$\alpha=\frac{\bar\beta+\beta}{2},\qquad \text{and}\quad (n-1)<m\quad \text{or}\quad (n-1)=m, \tag{153}$$

$$\alpha=\frac{\bar\beta+\beta}{2}+1,\qquad \text{and}\quad (n-1)<m\quad \text{or } (n-1)=m. \tag{154}$$

In case (153), orthonormality reduces (152) to

$$\mathcal{B}_{ij} = -T\int_0^\infty dz\ L_m^\alpha(z)e^{-z}z^{\frac{\bar\beta-1}{2}}\Big(z^{\frac{\bar\beta-1}{2}}+b_{(\bar\beta-3)}z^{\frac{\bar\beta-3}{2}}+b_{(\bar\beta-5)}z^{\frac{\bar\beta-5}{2}}+\ldots+b_0\Big), \tag{155}$$

for $(n-1)<m$, while for $(n-1)=m$ we have

$$\begin{aligned}\mathcal{B}_{ij} &= -b_{(\bar\beta+2n-1)}\int_0^\infty dz\ L_m^\alpha(z)e^{-z}z^\alpha z^m - \\ &\qquad T\int_0^\infty dz\ L_m^\alpha(z)e^{-z}z^\alpha\Big(z^{\frac{\bar\beta-1}{2}}+b_{(\bar\beta-3)}z^{\frac{\bar\beta-3}{2}}+b_{(\bar\beta-5)}z^{\frac{\bar\beta-5}{2}}+\ldots+b_0\Big). \quad (156)\end{aligned}$$

The first term in (155) reduces to

$$\begin{aligned}&-b_{(\bar\beta+2n-1)}\int_0^\infty dz\ L_m^\alpha(z)e^{-z}z^\alpha z^m = -b_{(\bar\beta+2n-1)}\int_0^\infty dz\ L_m^\alpha(z)e^{-z}z^\alpha\Big(\frac{m!}{(-1)^m}L_m^\alpha(z)\Big)\\ &\qquad = -b_{(\bar\beta+2n-1)}\frac{m!}{(-1)^m}h_m^\alpha = -b_{(\bar\beta+2n-1)}\frac{m!}{(-1)^m}\frac{\Gamma(\alpha+1+m)}{m!}\\ &\qquad = -b_{(\bar\beta+2n-1)}(-1)^m\Gamma(\alpha+1+m) = -a_n(-1)^m\Gamma(\alpha+1+m)\\ &= -\frac{(-1)^n}{n!}(-1)^m\Gamma(\alpha+1+m) = -\frac{(-1)^{m+1}}{(m+1)!}(-1)^m\Gamma(\alpha+1+m) = \frac{\Gamma(\alpha+m+1)}{\Gamma(m+2)}. \quad (157)\end{aligned}$$

Working out explicitly what the b_i's are in terms of Γ-functions and using [28]

$$\int_0^\infty dx\ e^{-x}x^{\gamma-1}L_n^\mu(x) = \frac{\Gamma[\gamma]\Gamma[1+\mu+n-\gamma]}{\Gamma[n+1]\Gamma[1+\mu-\gamma]}. \tag{158}$$

on each term in (155) gives us the result

$$\mathcal{B}_{ij} = -\sum_{i=0}^{n}(-1)^i\, 2^i \frac{\Gamma(\frac{\bar\beta+1}{2}+i)}{\Gamma(\frac{\bar\beta+1}{2})\Gamma(i+1)} \frac{\Gamma(n+1+\bar\alpha)}{\Gamma(n+1-i)\Gamma(\bar\alpha+1+i)} \times$$
$$\frac{1}{\Gamma(m+1)} \sum_{j=0}^{\frac{\bar\beta-1}{2}} \frac{\Gamma(\alpha-j)\Gamma(1+m+j)}{\Gamma(1+j)}\; 2^j \frac{\Gamma(\frac{\bar\beta+1}{2})}{\Gamma(\frac{\bar\beta+1}{2}-j)} (+) \frac{\Gamma(\alpha+m+1)}{\Gamma(m+2)}, \tag{159}$$

for

$$\frac{(\bar\beta+\beta)}{2} = \alpha, \quad (n-1) \le m, \text{ and } \bar\beta \text{ odd}, \tag{160}$$

The last term is only added in the case $(n-1) = m$.

In the case (154), only one term in the first line of (152) survives because of orthonormality. It reads

$$\begin{aligned} -b_{(\bar\beta+1)} \int_0^\infty dz\, L_m^\alpha(z) e^{-z} z^{\frac{\bar\beta+\beta}{2}} &= -b_{(\bar\beta+1)} \int_0^\infty dz\, L_m^\alpha(z) e^{-z} z^{\frac{\bar\beta+\beta}{2}+1} z^{-1} \\ &= -b_{(\bar\beta+1)} \int_0^\infty dz\, L_m^\alpha(z) e^{-z} z^\alpha z^{-1} = -b_{(\bar\beta+1)}\Gamma(\alpha). \end{aligned} \tag{161}$$

By (139) and (150) we have

$$\begin{aligned} b_{(\bar\beta+1)} &= \Big(a_1 + (\bar\beta+3)a_2 + (\bar\beta+5)(\bar\beta+3)a_3 + \ldots + (\bar\beta+2n-1)(\bar\beta+2n-3)\cdots(\bar\beta+3)a_n.\Big) \\ &= \frac{T-a_0}{(\bar\beta+1)}. \end{aligned} \tag{162}$$

The contribution from the second line in (152) is given by (159) without the (+) term and with the substitution $\alpha \to \alpha - 1$. Collecting the parts we get that for

$$\frac{\bar\beta+\beta}{2} + 1 = \alpha, \quad (n-1) \le m \quad , \text{and} \quad \bar\beta \text{ odd} \tag{163}$$

we have

$$\mathcal{B}_{ij} = \Big[\text{Righthand side of eq. (159) with } \alpha \to \alpha - 1 \text{ and } \textit{without} \text{ the term } (+)\Big] + \frac{T-a_0}{\bar\beta+1}\,\Gamma(\alpha), \tag{164}$$

where T is given by eq. (151).

References

[1] E.V.Shuryak and J.J.M.Verbaarschot, Nucl.Phys. **A560** (1993) 306.

[2] M.A.Halasz and J.J.M.Verbaarschot, Phys.Rev. **D52** (1995) 2563.

[3] A.Smilga and J.J.M.Verbaarschot, Phys.Rev. **D51** (1995) 829.

[4] J.J.M.Verbaarschot and I.Zahed, Phys.Rev.Lett. **73** (1994) 2288.

[5] J.Gasser and H.Leutwyler, Phys.Lett. **B184** (1987) 83.

[6] T.Banks and A.Casher, Nucl.Phys. **B169** (1980) 103.

[7] H.Leutwyler and A.Smilga, Phys.Rev. **D46** (1992) 5607.

[8] J.J.M.Verbaarschot and I.Zahed, Phys.Rev.Lett. **70** (1993) 3852.

[9] J.J.M.Verbaarschot, Phys.Rev.Lett. **72** (1994) 2531,
Nucl.Phys. **B426** (1994) 559.

[10] U.Magnea: *The ortogonal ensemble of random matrices and QCD in three dimensions*, hep-th/9907096.

[11] U.Magnea: *Three-dimensional QCD in the adjoint representation and random matrix theory*, hep-th/9912207.

[12] R.D.Pisarski, Phys.Rev. **D29** (1984) 2423.

[13] G.Akemann, P.H.Damgaard, U.Magnea and S.Nishigaki, Nucl.Phys. **B487** (1997) 721.

[14] M.K.Sener and J.J.M.Verbaarschot, Phys.Rev.Lett. **81** (1998) 248.

[15] J.J.M.Verbaarschot : *Universal Behavior in Dirac Spectra*, hep-th/9710114.

[16] J.Christiansen, Nucl.Phys. **B547** (1998) 329.

[17] P.H.Damgaard and S.Nishigaki, Phys.Rev. **D57** (1998) 5299.

[18] M. L. Mehta: *Random Matrices*, 2nd Edition, Academic Press (1991).

[19] H. Widom, J.Statist.Phys. **94** (1999) 347.

[20] T.Nagao and K.Slevin, J.Math. **34** (1993) 2075.

[21] G. Szego : *Orthogonal Polynomials*, 4th ed. Am. Math. Soc.,Providence, 1975.

[22] F.J.Dyson, Comm.Math. **19** (1970) 235.

[23] G.Mahoux and M.L Mehta, J.Phys.I. (France) **1** (1991) 1093.

[24] C.A.Tracy and H. Widom : *Correlation Functions, Cluster Functions and Spacing Distribibutions for Random Matrices*, solv-int/9804004.

[25] W.Magnus, F.Oberhettinger and R.P.Soni : *Formulars and Theorems for the Special Functions of Mathematical Physics*, 3.Edition, Springer-Verlag Berlin Heidelberg New York 1966.

[26] C.A.Craig and H. Widom, Commun.Math.Phys. **163** (1994) 33.

[27] J.J.M.Verbaarschot and A.Smilga, Phys.Rev. **D51** (1995) 829.

[28] A.Erdélyi (ed.) et al : *Higher Transcendental Functions*, Vol.II, McGraw-Hill (1953).

[29] T.Nagoa and S.M.Nishigaki, hep-th/0005077.